THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 67 NUMBER 1

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JANUARY 1960

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Treasurer, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
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(Devoted to the Interests of Collegiate Mathematics)

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Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For nonmembers the subscription price is $8.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June–July,
August–September, October, November, December.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.
Second-class postage paid at Menasha, Wisconsin.

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THE MATHEMATICAL ASSOCIATION OF AMERICA
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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 67
1960

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND BUFFALO, N. Y.

SOME CONCEPTS OF CONFORMAL GEOMETRY

PETER SCHERK, University of Toronto

There are several introductions to conformal geometry in the French and German literatures. However, there does not seem to exist a recent one in English. The present article cannot claim to be one. We only wish to develop a few basic notions of n-dimensional conformal geometry and to elaborate its connection with the projective geometry on the n-sphere and with $(n+2)$-dimensional pseudo-euclidean geometry.

As far as the author knows, general conformal geometry over an arbitrary field F has not yet been discussed. By means of a (generalized) stereographic projection it can be related to the geometry on an arbitrary regular n-dimensional quadric in a projective $(n+1)$-space over F. In its turn this geometry is related to a (generalized) euclidean geometry in a centro-affine $(n+2)$-space over F. The bilinear form defining this geometry will be regular but not necessarily definite. The first two geometries possess homogeneous orthogonal groups while the group of the last one is inhomogeneous. The transition to this last geometry is based on the well-known idea of providing the intersections of the quadric with the linear n-spaces with norms. The new space elements [$(n-1)$-quadrics plus norms] can be mapped one-one onto the vectors $\neq 0$ of affine $(n+2)$-space.

In the present note, only classical conformal geometry over the complex field is discussed. But the author has tried to use tools suitable for the more general situation indicated above. On the other hand he has also tried—in particular in the first sections—to motivate the introduction of conformal concepts from the point of view of elementary euclidean geometry.

The approach chosen in this paper is based on the stereographic projection of (most of) euclidean n-space E_n into the n-sphere in E_{n+1}. Insisting that this fundamental mapping be one-one, we are led in a natural way to imbed E_n into its conformal closure and E_{n+1} into projective $(n+1)$-space. The algebraic expression for this projection is obtained in several leisurely steps. An impatient reader is referred to formulas (3.21), (3.31), (3.32).

In the last sections inversions are discussed. They are shown to have most of the properties familiar from real 3-space. In particular the general conformal transformation can be decomposed into inversions. However, there are exceptions to Liouville's theorem in complex conformal spaces and not every conformal transformation will be the product of an inversion and a similarity.

1. Stereographic projection.

1.1. The stereographic mapping of the points

$$\xi = (\xi_1, \cdots, \xi_n) \tag{1.11}$$

in real euclidean n-space onto the points

$$\xi^* = (\xi_1^*, \cdots, \xi_n^*, \xi_{n+1}^*) \tag{1.12}$$

of the unit n-sphere S_n in euclidean $(n+1)$-space is well known.

Put

$$\xi_0 = \sum_1^n \xi_\lambda^2. \tag{1.13}$$

Then the image of the point ξ is the point

$$\xi^* = \frac{2}{\xi_0 + 1}(\xi_1, \cdots, \xi_n, \tfrac{1}{2}(\xi_0 - 1)). \tag{1.14}$$

Conversely, the point (1.12) of S_n has one and only one original point, *viz.*,

$$\xi = \frac{1}{1 - \xi_{n+1}^*}(\xi_1^*, \cdots, \xi_n^*). \tag{1.15}$$

However, there is one point on S_n without original points. This is the "north pole"

$$\mathfrak{v}^* = (0, \cdots, 0, 1) \tag{1.16}$$

of S_n. It is by introducing an ideal ("absolute") original point $\mathfrak{v}$ of $\mathfrak{v}^*$ that real euclidean n-space is completed to real conformal n-space.

1.2. Formula (1.14) can also be interpreted as a mapping of complex euclidean n-space E_n into the (from now on complex) unit sphere S_n. But now we have to note the following exceptions:

(i) No point of the $(n-1)$-dimensional hypersphere

$$\xi_0 + 1 = 0 \tag{1.21}$$

of radius i about the origin has an image point.

(ii) S_n intersects its tangent-hyperplane

$$\xi_{n+1}^* - 1 = 0 \tag{1.22}$$

at $\mathfrak{v}^*$ in an $(n-1)$-sphere S_{n-1}^* of radius zero ("null sphere") about $\mathfrak{v}^*$ (S_{n-1}^* can also be interpreted as a cone with the vertex $\mathfrak{v}^*$). By (1.15), no point of S_{n-1}^* has original points in E_n.

As a first step towards removing these exceptions we complete complex euclidean $(n+1)$-space to a complex projective $(n+1)$-space P_{n+1} by adding an "absolute hyperplane." This can be done by introducing homogeneous coordinates

$$\mathfrak{x}^* = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \tag{1.23}$$

connected with the inhomogeneous coordinates (1.12) of euclidean $(n+1)$-space through

$$\xi_\mu^* = x_\mu / x_{n+2} \qquad (\mu = 1, \cdots, n+1). \tag{1.24}$$

The absolute hyperplane is then given by

$$x_{n+2} = 0 \tag{1.25}$$

and the inhomogeneous equation $\sum_1^{n+1} \xi_\mu^{*2} - 1 = 0$ of S_n can now be rewritten in a homogeneous form

$$\sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2 = 0. \tag{1.26}$$

This way, S_n will be completed by adding the "absolute null sphere"

$$\sum_1^{n+1} x_\mu^2 = 0, \qquad x_{n+2} = 0. \tag{1.27}$$

In P_{n+1}, (1.27) is the intersection of S_n with the absolute hyperplane. It has the dimension $n-1$. By (1.24) and (1.14)

$$x_\lambda / x_{n+2} = \xi_\lambda^* = \xi_\lambda / \{\tfrac{1}{2}(\xi_0 + 1)\} \qquad (\lambda = 1, \cdots, n).$$

$$x_{n+1}/x_{n+2} = \xi_{n+1}^* = \{\tfrac{1}{2}(\xi_0 - 1)\}/\{\tfrac{1}{2}(\xi_0 + 1)\}. \tag{1.28}$$

This leads to the following expression for the stereographic projection of E_n into the complex unit sphere S_n in P_{n+1}:

$$\mathbf{x}^* = \rho(\xi_1, \cdots, \xi_n, \tfrac{1}{2}(\xi_0 - 1), \tfrac{1}{2}(\xi_0 + 1)). \tag{1.29}$$

Here $\rho \neq 0$ is an arbitrary scalar. By (1.27) and (1.13), the hypersphere (1.21) is mapped onto the absolute null sphere. Thus every point of E_n now has an image point.

1.3. In order to determine the original of a point (1.23) of S_n, we introduce the quantities

$$x_0 = \sum_1^n x_\lambda^2 \tag{1.31}$$

and

$$x_{-1} = x_{n+2} - x_{n+1}, \qquad x_{-2} = x_{n+2} + x_{n+1}. \tag{1.32}$$

Thus the equation (1.26) of S_n can be rewritten

$$x_0 - x_{-1} \cdot x_{-2} = 0 \tag{1.33}$$

and the completed tangent-hyperplane (1.22) of the north pole of S_n has the equation

$$x_{-1} = 0. \tag{1.34}$$

If the point $\mathbf{x}^*$ of S_n is the image of the point ξ of E_n, (1.29) and (1.31) yield

(1.35) $$x_0 = \rho^2\xi_0, \qquad x_{-1} = \rho, \qquad x_{-2} = \rho\xi_0.$$

From (1.28) we obtain the following version of (1.15)

(1.36) $$\xi = (1/x_{-1})(x_1, \cdots, x_n).$$

Similarly, (1.35) implies

(1.37) $$\rho = x_{-1}, \qquad \xi_0 = x_0/x_{-1}^2 = x_{-2}/x_{-1}.$$

1.4. Equation (1.36) breaks down if $x_{-1}=0$, *i.e.*, for points $\mathfrak{x}^*$ on the completed null sphere S^*_{n-1} of **1.2.** As a first step towards eliminating these exceptions we note that (1.36) suggests introducing $x_1, \cdots, x_n$, and x_{-1} as homogeneous coordinates in E_n. However we add one redundant coordinate x_{-2} which is connected with the others through

(1.41) $$x_0 = \sum_1^n x_\lambda^2 = x_{-1}x_{-2};$$

cf. (1.31) and (1.33). Thus the point ξ determines its homogeneous coordinate vector

(1.42) $$\mathfrak{x} = \rho(\xi_1, \cdots, \xi_n, 1, \xi_0).$$

It obviously satisfies (1.35) and (1.41); in particular

(1.43) $$x_{-1} = \rho \neq 0.$$

Conversely assume (1.41) and $x_{-1}\neq 0$. Then the vector

(1.44) $$\mathfrak{x} = (x_1, \cdots, x_n, x_{-1}, x_{-2})$$

determines the point ξ in E_n through (1.36).

If homogeneous coordinates are used both in E_n and on S_n, the formulas for stereographic projection become particularly simple. Combining (1.36) with (1.29) and choosing $\rho = x_{-1}$ in the latter, we find that the point (1.44) in E_n determines the point (1.23) on S_n, where

(1.45) $$x_{n+1} = \tfrac{1}{2}(x_{-2} - x_{-1}), \qquad x_{n+2} = \tfrac{1}{2}(x_{-2} + x_{-1})$$

are obtained from (1.32).

1.5. It is now easy to enlarge E_n to complex conformal n-space C_n by interpreting each homogeneous vector $\mathfrak{x}\neq 0$ which satisfies (1.41) as a point of C_n. Thus two such vectors are associated with the same point if and only if they are proportional. The "finite points" of C_n, *i.e.*, the points of E_n, are characterized by $x_{-1}\neq 0$. The "absolute points" of C_n which we have added are those with coordinate vectors

(1.51) $$\mathfrak{x} = (x_1, \cdots, x_n, 0, x_{-2}).$$

By (1.41) they satisfy

$$x_0 = \sum_1^n x_\lambda^2 = 0. \tag{1.52}$$

Thus x_{-2} is no longer determined by the other coordinates. We call the set of the absolute points the absolute null sphere of C_n. Obviously it contains exactly one real point, *viz.*,

$$\mathbf{n} = (0, \cdots, 0, 0, 1). \tag{1.53}$$

Extending **1.4** we now define the stereographic projection of the point (1.44) of C_n as the point (1.23) of S_n, where x_{-1}, x_{-2} and x_{n+1}, x_{n+2} are connected by (1.32) and (1.45). We have finally disposed of the exceptions noted in **1.2** and C_n is mapped one-one onto S_n.

The image of a point (1.51) is a point of S_n satisfying

$$x_{-1} = x_{n+2} - x_{n+1} = 0. \tag{1.54}$$

Thus the absolute null sphere of C_n is mapped onto the absolute null sphere S_{n-1}^* on S_n discussed in **1.2.** In particular the image of the real absolute point $\mathbf{n}$ is the north pole of S_n

$$\mathbf{n}^* = (0, \cdots, 0, 1, 1). \tag{1.55}$$

2. Spheres in C_n.

2.1. An $(n-1)$-sphere S_{n-1} in complex euclidean n-space E_n is given by an equation

$$a_{-1}\xi_0 - 2\sum_1^n a_\lambda \xi_\lambda + a_{-2} = 0. \tag{2.11}$$

By (1.36) and (1.37), S_{n-1} has the equation

$$\sum_1^n a_\lambda \cdot x_\lambda - \tfrac{1}{2}(a_{-2}x_{-1} + a_{-1}x_{-2}) = 0 \tag{2.12}$$

in homogeneous coordinates. We call

$$\mathbf{a} = (a_1, \cdots, a_n, a_{-1}, a_{-2}) \tag{2.13}$$

the homogeneous vector of S_{n-1}. It is determined by S_{n-1} up to a proportionality factor. Every vector $\mathbf{a}$ which is not a multiple of $\mathbf{n} = (0, \cdots, 0, 0, 1)$ determines an $(n-1)$-sphere.

The $(n-1)$-planes are contained among the $(n-1)$-spheres. They are characterized by

$$a_{-1} = 0. \tag{2.14}$$

Let $a_{-1} \neq 0$. By (2.11), the center of S_{n-1} has the inhomogeneous coordinates

$$\alpha = (1/a_{-1})(a_1, \cdots, a_n). \tag{2.15}$$

Thus by (1.42), it has the homogeneous coordinates

(2.16) $$(a_1, \cdots, a_n, a_{-1}, a_0/a_{-1}),$$

where

(2.17) $$a_0 = \sum_1^n a_\lambda^2.$$

By (2.11), the square of the radius of S_{n-1} is

(2.18) $$r^2 = (a_0 - a_{-1}a_{-2})/a_{-1}^2.$$

Let $a_{-1} \neq 0$. Comparing (2.18), (2.16), (2.13), and (2.12) we see that the following properties of S_{n-1} are equivalent:

(i) S_{n-1} is a null sphere;
(ii) $a_0 - a_{-1}a_{-2} = 0$;
(iii) $\mathbf{a}$ is the center of S_{n-1};
(iv) $\mathbf{a} \in S_{n-1}$.

2.2. The above discussion is readily extended to conformal n-space C_n. On account of (2.14), the $(n-1)$-planes can now be characterized as the $(n-1)$-spheres through $\mathbf{n}$.

The absolute points (1.51) of C_n satisfy the equation $x_{-1} = 0$. Comparison with (2.12) shows that the absolute null sphere can actually be interpreted as an $(n-1)$-sphere with the vector $\mathbf{n}$. As it satisfies (2.14), formula (2.18) becomes meaningless. However, conditions (ii) and (iv) of **2.1** hold for $\mathbf{n}$. Multiplying (2.16) by a_{-1} before substituting $\mathbf{n}$, we see that $\mathbf{n}$ may be interpreted as the centre of the absolute null sphere.

Through the introduction of this sphere, we achieve a one-one correspondence between the $(n-1)$-spheres in C_n and the homogeneous vectors $\mathbf{a} \neq 0$.

2.3. The last remark suggests an interpretation of the $(n-1)$-spheres S_{n-1} of C_n as the points of a projective $(n+1)$-space. We readily construct an explicit mapping.

Suppose S_{n-1} is given by (2.12). Its stereographic projection S_{n-1}^* was the locus of the points

(2.31) $$\mathbf{x}^* = (x_1, \cdots, x_n, x_{n+1}, x_{n+2})$$

on the unit sphere S_n in complex projective $(n+1)$-space P_{n+1}. Here x_{n+1} and x_{n+2} were determined by (1.45) or (1.32). Thus the points (2.31) satisfy

$$\sum_1^n a_\lambda \cdot x_\lambda - \tfrac{1}{2}[a_{-2}(x_{n+2} - x_{n-1}) + a_{-1}(x_{n+2} + x_{n+1})] = 0$$

or

(2.32) $$\sum_1^n a_\lambda x_\lambda + a_{n+1}x_{n+1} - a_{n+2}x_{n+2} = 0,$$

where

$$a_{n+1} = \tfrac{1}{2}(a_{-2} - a_{-1}), \qquad a_{-1} = a_{n+2} - a_{n+1},$$

(2.33) or

$$a_{n+2} = \tfrac{1}{2}(a_{-2} + a_{-1}), \qquad a_{-2} = a_{n+2} + a_{n+1}.$$

Equation (2.32) determines an n-plane in P_{n+1}. Thus its intersection S^*_{n-1} with S_n is an $(n-1)$-sphere. Obviously, the point

$$\mathbf{a}^* = (a_1, \cdots, a_n, a_{n+1}, a_{n+2}) \tag{2.34}$$

is the pole with respect to S_n of the n-plane (2.32). Thus

$$S_{n-1} \leftrightarrow \mathbf{a}^* \tag{2.35}$$

is a one-one correspondence between the $(n-1)$-spheres of C_n and the points of P_{n+1}.

By means of (2.32) and (2.33) we verify once more that the stereographic projection of the absolute null sphere of C_n is the null sphere in which S_n intersects its tangent-hyperplane at its north pole $\mathbf{n}^* = (0, \cdots, 0, 1, 1)$. Furthermore, S_{n-1} is a hyperplane if and only if

$$\mathbf{n}^* \in S^*_{n-1}. \tag{2.36}$$

S_{n-1} is a null sphere if and only if

$$\mathbf{a}^* \in S_n. \tag{2.37}$$

3. Conformal and pseudo-euclidean metrics.

3.1. We have interpreted

$$(x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \tag{3.11}$$

as the homogeneous coordinate vector of a point $\mathbf{x}^*$ in complex projective $(n+1)$-space P_{n+1}. But (3.11) can also be interpreted as the inhomogeneous coordinate vector of a point $\bar{\mathbf{x}}$ in complex affine $(n+2)$-space A_{n+2}. Then if $\bar{\mathbf{x}}$ is distinct from the origin O of A_{n+2}, it determines a point $\mathbf{x}^*$ in P_{n+1}; and two points of A_{n+2} determine the same point in P_{n+1} if and only if they lie on the same straight line through O. Thus with each point $\mathbf{x}^* \in P_{n+1}$ all the points $\neq O$ of a line through O are associated. If $\mathbf{x}^*$ lies in S_n, they satisfy (1.26). Thus S_n corresponds to the $(n+1)$-cone K_{n+1}: $\sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2 = 0$ in A_{n+2}.

We now map the point

$$\bar{\mathbf{x}} = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \neq 0 \tag{3.12}$$

of K_{n+1} onto the point

$$\mathbf{x} = (x_1, \cdots, x_n, x_{-1}, x_{-2}) \tag{3.13}$$

of C_n with the *weight* x_{-1}:

(3.14) $$\bar{\mathbf{x}} \to \mathbf{x},\ x_{-1};$$

cf. (1.32). Given a point $\mathbf{x} \in C_n$ and a weight ρ. We require a point $\bar{\mathbf{x}}$ which is mapped by (3.14) onto $\mathbf{x}$, ρ. Unless neither or both of the numbers ρ and $x_{-1} = x_{n+2} - x_{n+1}$ vanish, this problem has no solution.

If x_{-1} and ρ are both distinct from zero, we can multiply the coordinates of $\mathbf{x}$ by exactly one factor, *viz.*, ρ/x_{-1} such that

(3.15) $$\rho = x_{-1}$$

for the new coordinates of $\mathbf{x}$. Thus we have exactly one solution $\bar{\mathbf{x}}$.

If both x_{-1} and ρ vanish, there is a straight line through O in the $(n+1)$-plane

(3.16) $$x_{-1} = x_{n+2} - x_{n+1} = 0$$

such that all the points of this line are solutions.

3.2. From now on, the same letter $\mathbf{x}^*$ will be used to denote points in P_{n+1} and in A_{n+2}. Furthermore, vectors with $n+2$ components will be considered as column vectors, with primes indicating transposition, *e.g.*,

(3.21) $$\mathbf{x}' = (x_1, \cdots, x_n, x_{-1}, x_{-2}), \qquad \mathbf{x}^{*\prime} = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}).$$

I_n is the $n \times n$ unit matrix. The matrix

(3.22) $$J = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & 1 & 0 \\ 0 & 0 & -1 \end{array}\right)$$

defines a pseudo-euclidean scalar product

(3.23) $$\mathbf{x}^{*\prime} J \mathbf{y}^* = \sum_1^n x_\lambda y_\lambda + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}$$

in A_{n+2}. The locus of the points $\mathbf{x}^*$ with

(3.24) $$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0$$

is an $(n+1)$-plane through O. The pseudo-euclidean norm of the vector $\mathbf{x}^*$ is

(3.25) $$\|\mathbf{x}^*\| = \mathbf{x}^{*\prime} J \mathbf{x}^* = \sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2.$$

Thus the points of K_{n+1} are characterized by

(3.26) $$\|\mathbf{x}^*\| = 0.$$

In particular, $\|\mathbf{x}^*\|$ can be zero (or negative) for a real vector $\mathbf{x}^* \neq 0$.

The pseudo-euclidean distance δ between two points $\mathbf{x}^*$ and $\mathbf{y}^*$ of A_{n+2} is any root of

$$\delta^2 = \|\mathbf{x}^* - \mathbf{y}^*\| = \|\mathbf{x}^*\| + \|\mathbf{y}^*\| - 2\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.27}$$

If both $\mathbf{x}^*$ and $\mathbf{y}^*$ lie in K_{n+1}, their norms vanish and we have

$$\delta^2 = -2\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.28}$$

Let $\|\mathbf{a}^*\| \neq 0$, $\|\mathbf{b}^*\| \neq 0$. Then we define the pseudo-euclidean angle ϕ between the $(n+1)$-planes (3.24) and

$$\mathbf{b}^{*\prime}J\mathbf{y}^* = 0 \tag{3.29}$$

through

$$\cos\phi = (\mathbf{a}^{*\prime}J\mathbf{b}^*)/\sqrt{(\|\mathbf{a}^*\| \cdot \|\mathbf{b}^*\|)}. \tag{3.210}$$

The planes (3.24) and (3.29) or the vectors $\mathbf{a}^*$ and $\mathbf{b}^*$ are called pseudo-orthogonal if

$$\mathbf{a}^{*\prime}J\mathbf{b}^* = 0. \tag{3.211}$$

We may call $\mathbf{a}^*$ a pseudo-normal vector of the $(n+1)$-plane (3.24).

The preceding definitions can be adapted to P_{n+1} with few changes. Equation (3.24) then represents an n-plane, (3.26) characterizes the points of S_n, and (3.210) can still be used to define the angle between two n-planes.

3.3. Through (1.32), every point $\mathbf{x}^*$ in P_{n+1} (or in A_{n+2}) is mapped onto a spherical vector $\mathbf{x}$ in C_n. It is convenient to rewrite (1.32) in matrix form

$$\mathbf{x} = U\mathbf{x}^*, \tag{3.31}$$

where

$$U = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -1 & 1 \\ 0 & 1 & 1 \end{array}\right), \qquad U^{-1} = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right). \tag{3.32}$$

Put

$$G = U^{-1}JU^{-1} = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{array}\right). \tag{3.33}$$

The conformal scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ is the number

$$\mathbf{x}'G\mathbf{y} = \sum_1^n x_\lambda y_\lambda - \tfrac{1}{2}(x_{-1}y_{-2} + x_{-2}y_{-1}). \tag{3.34}$$

Assume (3.31) and

$$\mathbf{y} = U\mathbf{y}^*. \tag{3.35}$$

By (3.33) the conformal and pseudo-euclidean scalar products are related through

$$\mathbf{x}'G\mathbf{y} = \mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.36}$$

The conformal norm of $\mathbf{x}$ is

$$\|\mathbf{x}\| = \mathbf{x}'G\mathbf{x} = \sum_1^n x_\lambda^2 - x_{-1}x_{-2} = x_0 - x_{-1}x_{-2}. \tag{3.37}$$

By (3.36),

$$\|\mathbf{x}\| = \|\mathbf{x}^*\|. \tag{3.38}$$

The points (or null spheres) in C_n are characterized by

$$\|\mathbf{x}\| = 0. \tag{3.39}$$

Hence

$$\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\| - 2\mathbf{x}'G\mathbf{y} = -2\mathbf{x}'G\mathbf{y} \tag{3.310}$$

if $\mathbf{x}$ and $\mathbf{y}$ are points in C_n; *cf.* (3.28).

3.4. Given two finite points $\mathbf{x}$ and $\mathbf{y}$ in C_n with the weights x_{-1} and y_{-1}, respectively; *cf.* **3.1.** Let $\xi(\mathfrak{n})$ denote the inhomogeneous coordinates of $\mathbf{x}$ (of $\mathbf{y}$). Then by (1.13), (1.37), and (1.36)

$$\begin{aligned}(\xi - \mathfrak{n})^2 &= \sum_1^n (\xi_\lambda - \eta_\lambda)^2 = \sum_1^n \xi_\lambda^2 + \sum_1^n \eta_\lambda^2 - 2\sum_1^n \xi_\lambda\eta_\lambda \\ &= \xi_0 + \eta_0 - 2\sum_1^n \xi_\lambda\eta_\lambda = \frac{x_{-2}}{x_{-1}} + \frac{y_{-2}}{y_{-1}} - 2\sum_1^n \frac{x_\lambda y_\lambda}{x_{-1}y_{-1}} \\ &= -\frac{2}{x_{-1}y_{-1}}\left[\sum_1^n x_\lambda y_\lambda - \tfrac{1}{2}(x_{-1}y_{-2} + x_{-2}y_{-1})\right].\end{aligned}$$

Thus by (3.34) and (3.310)

$$x_{-1}y_{-1}(\xi - \mathfrak{n})^2 = -2\mathbf{x}'G\mathbf{y} = \|\mathbf{x} - \mathbf{y}\|. \tag{3.41}$$

3.5. By (2.12), the locus of the points $\mathbf{x}$ in C_n which satisfy an equation

$$\mathbf{a}'G\mathbf{x} = 0 \tag{3.51}$$

is an $(n-1)$-sphere. It is proper if and only if $a_{-1} \neq 0$. By (2.18) and (3.37), its radius r is given by

$$r^2 = (a_0 - a_{-1}a_{-2})/a_{-1}^2 = \|\mathbf{a}\|/a_{-1}^2. \tag{3.52}$$

Now let

(3.53) $$\mathbf{b}'G\mathbf{x} = 0$$

be a second proper $(n-1)$-sphere and let s be its radius. Let α and β denote the inhomogeneous coordinate vectors of the centers of the $(n-1)$-spheres (3.51) and (3.53) respectively. By (3.41), (2.16), (3.34) and (3.37)

$$\begin{aligned} a_{-1}b_{-1}(\alpha-\beta)^2 &= -2\left[\sum_1^n a_\lambda b_\lambda - \frac{1}{2}\left(a_{-1}\frac{b_0}{b_{-1}} + b_{-1}\frac{a_0}{a_{-1}}\right)\right] \\ &= -2\left[\sum_1^n a_\lambda b_\lambda - \tfrac{1}{2}(a_{-1}b_{-2} + a_{-2}b_{-1})\right] + a_{-1}\left(\frac{b_0}{b_{-1}} - b_{-2}\right) \\ &\quad + b_{-1}\left(\frac{a_0}{a_{-1}} - a_{-2}\right) \\ &= -2\mathbf{a}'G\mathbf{b} + \frac{a_{-1}}{b_{-1}}\|\mathbf{b}\| + \frac{b_{-1}}{a_{-1}}\|\mathbf{a}\|. \end{aligned}$$

Thus (3.52) leads to the following generalization of (3.41)

(3.54) $$a_{-1}b_{-1}[(\alpha-\beta)^2 - r^2 - s^2] = -2\mathbf{a}'G\mathbf{b}.$$

If our two $(n-1)$-spheres are real and have a real intersection, the angle ϕ between them is determined by

$$(\beta-\alpha)^2 = r^2 + s^2 - 2rs\cos\phi.$$

Solving for ϕ and substituting (3.54) and (3.52), we obtain

(3.55) $$\cos\phi = (\mathbf{a}'G\mathbf{b})/\sqrt{(\|\mathbf{a}\|\cdot\|\mathbf{b}\|)}.$$

If (3.51) is the equation of a real finite $(n-1)$-plane, the distance of a point $\mathbf{z}$ from it is equal to the absolute value of

(3.56) $$(\mathbf{a}'G\mathbf{z})/(z_{-1}\cdot\|\mathbf{a}\|).$$

This readily implies that (3.55) remains valid if one (or both) of our $(n-1)$-spheres degenerate into $(n-1)$-planes. We may therefore define the angle ϕ between any two nonnull spheres through (3.55).

Two arbitrary $(n-1)$-spheres (3.51) and (3.53) will be called orthogonal if

(3.57) $$\mathbf{a}'G\mathbf{b} = 0.$$

3.6. Let

(3.61) $$\mathbf{a}^* = U^{-1}\mathbf{a}, \qquad \mathbf{b}^* = U^{-1}\mathbf{b}.$$

By(3.36) and (3.38),

(3.62) $$(\mathbf{a}'G\mathbf{b})/\sqrt{(\|\mathbf{a}\|\cdot\|\mathbf{b}\|)} = (\mathbf{a}^{*\prime}J\mathbf{b}^*)/\sqrt{(\|\mathbf{a}^*\|\cdot\|\mathbf{b}^*\|)}.$$

As the right-hand expression is equal to the cosine of the angles between the $(n-1)$-spheres

$$\mathbf{a}^{*\prime}J\mathbf{x}^* = 0, \qquad \mathbf{b}^{*\prime}J\mathbf{x}^* = 0, \tag{3.63}$$

on S_n, (3.62) states the well-known invariance of angles under stereographic projection.

Through $\mathbf{a}^* = U^{-1}\mathbf{a}$, the $(n-1)$-spheres $\mathbf{a}'G\mathbf{x}=0$ of C_n are also mapped one-one onto the $(n+1)$-planes $\mathbf{a}^{*\prime}J\mathbf{x}^*=0$ in A_{n+2} through the origin. Equation (3.62) also states that the pseudo-euclidean angle between two such $(n+1)$-planes is equal to the angle between the corresponding $(n-1)$-spheres in C_n.

4. The conformal group.

4.1. Let

$$\mathbf{y}^* = T^*\mathbf{x}^* \tag{4.11}$$

be any linear transformation of pseudo-euclidean $(n+2)$-space A_{n+2} into itself. Then

$$\mathbf{x} = U\mathbf{x}^*, \qquad \mathbf{y} = U\mathbf{y}^* \tag{4.12}$$

implies

$$\mathbf{y} = UT^*U^{-1}\mathbf{x}. \tag{4.13}$$

Thus T^* determines a linear transformation

$$T = UT^*U^{-1} \tag{4.14}$$

of the spherical vectors in conformal n-space C_n. By (3.32), $U = U'$. Hence (4.14) implies

$$T' = U^{-1}T^{*\prime}U. \tag{4.15}$$

4.2. The linear transformation T^* maps K^{n+1} into itself if and only if

$$T^{*\prime}JT^* = \rho J \tag{4.21}$$

is a multiple of J. This implies

$$(T^*\mathbf{x}^*)'J\cdot(T^*\mathbf{y}^*) = \rho\cdot\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{4.22}$$

Thus T^* then multiplies all the scalar products (in particular, all the norms) in A_{n+2} by the same factor. Hence the pseudo-euclidean angle is invariant under T^*. We may interpret T^* as a pseudo-euclidean similarity about the origin of A_{n+2}. The transformations T^* with $\rho\neq 0$ form a group Σ^*.

If (4.21) holds with $\rho=1$, the pseudo-euclidean scalar products, norms and distances are invariant under T^*. Conversely, if all the norms are invariant under T^*, then

$$T^{*\prime}JT^* = J. \tag{4.23}$$

These T^*'s form a subgroup Ω^* of Σ^*, the pseudo-orthogonal group of A_{n+2}.

Let T^* satisfy (4.21) with $\rho\neq 0$; thus T^* is regular. Let

(4.24) $$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0$$

be the $(n+1)$-space through O in A_{n+2} with the pseudo-normal vector $\mathbf{a}^*$. By (4.24) and (4.21),

$$(T^*\mathbf{a}^*)' J (T^*\mathbf{x}^*) = \mathbf{a}^{*\prime}(T^{*\prime} J T^*)\mathbf{x}^* = \rho \mathbf{a}^{*\prime} J \mathbf{x}^* = 0.$$

Thus T^* maps (4.24) onto the $(n+1)$-space through O with the pseudo-normal vector $T^*\mathbf{a}^*$.

4.3. Through (4.14), Σ^* is mapped onto the full conformal group Σ. Since

$$T^* = U^{-1} T U, \qquad T^{*\prime} = U T' U^{-1},$$

(4.21) is equivalent to

$$(U T' U^{-1}) J (U^{-1} T U) = \rho J.$$

Thus by (3.33), Σ consists of those T which satisfy

(4.31) $$T' G T = \rho G, \qquad \rho \neq 0.$$

This implies

(4.32) $$(T\mathbf{a})' G (T\mathbf{b}) = \rho \cdot \mathbf{a}' G \mathbf{b}.$$

Hence T maps $(n-1)$-spheres in C_n onto $(n-1)$-spheres, null spheres onto null spheres and keeps the angle between $(n-1)$-spheres unchanged.

The image of the pseudo-orthogonal group Ω^* is the proper conformal group Ω. It consists of those $T \in \Sigma$ which keep scalar products invariant. Thus they are characterized by

(4.33) $$T' G T = G$$

or by

(4.34) $$\|T\mathbf{x}\| = \|\mathbf{x}\|.$$

The spherical vectors $\mathbf{a}$ and $\lambda\mathbf{a}$ determine the same $(n-1)$-sphere provided, perhaps, with different norms. Similarly, with $\mathbf{x}$, $\lambda\mathbf{x}$ is the vector of a point. The two points coincide but may have distinct weights x_{-1} and λx_{-1}. If we interpret $\mathbf{a}$ as a homogeneous spherical vector, then T and λT will induce the same transformation. It is only the weights of the points and the norms of the $(n-1)$-spheres of C_n which are transformed differently. Thus, if we abstract from norms and weights, the group of C_n will not be Σ but the factor group of Σ with respect to the invariant subgroup formed by the multiples λI_{n+2} of the identity ($\lambda \neq 0$). The complex field being quadratically closed, this factor group is isomorphic to the factor group of Ω with respect to the subgroup consisting of I_{n+2} and $-I_{n+2}$.

4.4. Put

(4.41) $$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$G = \begin{pmatrix} I_n & 0 \\ 0 & -\frac{1}{2}D \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & -2D \end{pmatrix}. \tag{4.42}$$

Let

$$T = \left(\begin{array}{ccc|cc} t_{11} & \cdots & t_{1n} & t_{1,-1} & t_{1,-2} \\ \vdots & & \vdots & \vdots & \vdots \\ t_{n1} & \cdots & t_{nn} & t_{n,-1} & t_{n,-2} \\ \hline t_{-11} & \cdots & t_{-1n} & t_{-1,-1} & t_{-1,-2} \\ t_{-21} & \cdots & t_{-2n} & t_{-2,-1} & t_{-2,-2} \end{array}\right) = \left(\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right) \tag{4.43}$$

be a transformation in Σ. Then by (4.31) and (4.42)

$$T^{-1} = \frac{1}{\rho} G^{-1} T' G = \frac{1}{\rho} \begin{pmatrix} I_n & 0 \\ 0 & -2D \end{pmatrix} \begin{pmatrix} T'_{11} & T'_{21} \\ T'_{12} & T'_{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -\frac{1}{2}D \end{pmatrix}$$

or

$$T^{-1} = \frac{1}{\rho} \begin{pmatrix} T'_{11} & -\frac{1}{2}T'_{21}D \\ -2DT'_{12} & DT'_{22}D \end{pmatrix}$$

$$= \frac{1}{\rho} \left(\begin{array}{ccc|cc} t_{11} & \cdots & t_{n1} & -\frac{1}{2}t_{-21} & -\frac{1}{2}t_{-11} \\ \vdots & & \vdots & \vdots & \vdots \\ t_{1n} & \cdots & t_{nn} & -\frac{1}{2}t_{-2n} & -\frac{1}{2}t_{-1n} \\ \hline -2t_{1,-2} & \cdots & -2t_{n,-2} & t_{-2,-2} & t_{-1,-2} \\ -2t_{1,-1} & \cdots & -2t_{n,-1} & t_{-2,-1} & t_{-1,-1} \end{array}\right). \tag{4.44}$$

The transformation T maps the spherical vector $\mathbf{x}$ onto the spherical vector

$$\mathbf{y} = T\mathbf{x}. \tag{4.45}$$

Points are mapped onto points.

If $\mathbf{x}$ and $\mathbf{y}$ are finite points, let ξ and $\mathfrak{n}$ denote their inhomogeneous coordinate vectors. Then by (4.45) and (4.43)

$$\eta_k = \frac{y_k}{y_{-1}} = \frac{\sum_1^n t_{k\lambda}x_\lambda + t_{k,-1}x_{-1} + t_{k,-2}x_{-2}}{\sum_1^n t_{-1\lambda}x_\lambda + t_{-1,-1}x_{-1} + t_{-1,-2}x_{-2}}.$$

Dividing numerator and denominator by x_{-1}, we obtain, on account of (1.36) and (1.37),

$$\eta_k = \frac{\sum_1^n t_{k\lambda}\xi_\lambda + t_{k,-1} + t_{k,-2}\xi_0}{\sum_1^n t_{-1\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0}. \tag{4.46}$$

The denominator is essentially the weight of the image point $\mathfrak{n}$:

$$y_{-1} = x_{-1}\left(\sum_1^n t_{-1\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0\right). \tag{4.47}$$

By (4.43) or (4.47), the original of the absolute null sphere is the $(n-1)$-sphere

$$y_{-1} = \sum_1^n t_{-1,\lambda}x_\lambda + t_{-1,-1}x_{-1} + t_{-1,-2}x_{-2} = 0 \tag{4.48}$$

or

$$y_{-1}/x_{-1} = \sum_1^n t_{-1,\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0 = 0. \tag{4.49}$$

Replacing T by T^{-1} and (4.43) by (4.44), we obtain the image of the absolute null sphere

$$-\tfrac{1}{2}\rho x_{-1} = \sum_1^n t_{\lambda,-2}y_\lambda - \tfrac{1}{2}(t_{-2,-2}y_{-1} + t_{-1,-2}y_{-2}) = 0 \tag{4.410}$$

or

$$-\tfrac{1}{2}\rho x_{-1}/y_{-1} = \sum_1^n t_{\lambda,-2}\eta_\lambda - \tfrac{1}{2}(t_{-2,-2} + t_{-1,-2}\eta_0) = 0. \tag{4.411}$$

4.5. By (4.48), (4.410), (4.47), the following properties of the conformal transformation

$$\mathbf{y} = T\mathbf{x} \tag{4.51}$$

are equivalent:

(4.52) The equation $x_{-1}=0$ is invariant under (4.51);
(4.53) T maps the absolute null sphere onto itself;
(4.54) $t_{-11}=t_{-12}= \cdots =t_{-1n}=t_{-1,-2}=0$;
(4.55) $t_{1,-2}=t_{2,-2}= \cdots =t_{n,-2}=t_{-1,-2}=0$;
(4.56) T multiplies the weights of all the finite points by the same factor $t_{-1,-1}$.

Every transformation in Σ multiplies the conformal norm by a constant factor. Hence (4.56) and (3.41) imply

(4.57) T is a similarity.

Suppose the transformation $T\in\Sigma$ is a projectivity. Then it maps $(n-1)$-planes into $(n-1)$-planes. Since they are determined by vectors $\mathbf{a}$ with $a_{-1}=0$, T must satisfy (4.52) and therefore also (4.57). We thus have:

(i) The only projectivities in Σ are similarities;
(ii) (4.57) is equivalent to each of (4.52)–(4.56).

Now let $T\in\Sigma$ be a similarity. By (4.54) and (4.55)

$$T=\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & t_{-1,-1} & 0 \\ \mathbf{v}' & t_{-2,-1} & t_{-2,-2} \end{pmatrix}, \qquad \tau=\begin{pmatrix} t_{1,-1} \\ \cdot \\ \cdot \\ \cdot \\ t_{n,-1} \end{pmatrix}, \qquad \mathbf{v}=\begin{pmatrix} t_{-21} \\ \cdot \\ \cdot \\ \cdot \\ t_{-2n} \end{pmatrix}.$$

Since $T\in\Sigma$, we may apply (4.31) obtaining

$$\begin{pmatrix} T_{11}'T_{11} & T_{11}'\tau-\frac{1}{2}t_{-1,-1}\mathbf{v} & 0 \\ \tau T_{11}-\frac{1}{2}t_{-1,-1}\mathbf{v} & \tau'\tau-t_{-1,-1}t_{-2,-1} & -\frac{1}{2}t_{-1,-1}t_{-2,-2} \\ 0 & -\frac{1}{2}t_{-1,-1}t_{-2,-2} & 0 \end{pmatrix}=\rho\begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

This yields

$$T=\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & t_{-1,-1} & 0 \\ 2\tau'T_{11}/t_{-1,-1} & \tau'\tau/t_{-1,-1} & \rho/t_{-1,-1} \end{pmatrix},$$

where

$$T_{11}'T_{11}=\rho I_n, \qquad \rho\neq 0, \tag{4.58}$$

or after a slight change of notation

$$T=\sigma\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & 1 & 0 \\ 2\tau'T_{11} & \tau'\tau & \rho \end{pmatrix}, \tag{4.59}$$

where $\rho\sigma\neq 0$ and where T_{11} still satisfies (4.58). Thus every similarity $T\in\Sigma$ admits a representation (4.59). Conversely, if T is given by (4.59) and (4.58), it satisfies (4.31) and therefore belongs to Σ. Since it also satisfies (4.54) it is a similarity. Thus (4.59)–(4.58) is another characterization of the similarities in Σ.

Now let

$$\eta_k=\sum_1^n t_{k\lambda}\xi_\lambda+t_{k,-1} \qquad (k=1,\cdots,n) \tag{4.510}$$

be an arbitrary similarity. Put

$$T_{11}=(t_{k\lambda}), \qquad \tau'=(t_{1,-1},\cdots,t_{n,-1}), \qquad \sigma \text{ arbitrary},$$

and define T through (4.59). Thus (4.58) holds. By (4.46), (4.510) is the inhomogeneous version of $\mathbf{y} = T\mathbf{x}$ and we have

Every similarity belongs to Σ and the multiplier $t_{-1,-1}$ of (4.56) can be chosen arbitrarily.

The similarity (4.510) is an orthogonality if and only if $\rho = 1$ in (4.58)–(4.59). Hence any of the conditions (4.52)–(4.57) together with

$$t_{-1,-1} = t_{-2,-2} \tag{4.511}$$

characterizes the orthogonalities.

The transformation (4.510) is a translation if and only if $T_{11} = I_n$. Since $\rho = 1$, we obtain from (4.59): the transformation T in Σ is a translation if and only if

$$T = \sigma \begin{pmatrix} I_n & \tau & 0 \\ 0 & 1 & 0 \\ 2\tau' & \tau'\tau & 1 \end{pmatrix}. \tag{4.512}$$

5. Symmetries in A_{n+2}.

5.1. Let

$$\alpha^*: \qquad \mathbf{a}^{*\prime} J \mathbf{x}^* = 0$$

be an $(n+1)$-space through O in $(n+2)$-dimensional pseudo-euclidean space A_{n+2}. We assume

$$\|\mathbf{a}^*\| = \mathbf{a}^{*\prime} J \mathbf{a}^* \neq 0. \tag{5.11}$$

Thus the pseudo-normal vector $\mathbf{a}^*$ of α^* does not lie in $\mathbf{a}^{*\prime}$ and every vector $\mathbf{x}^*$ permits a unique decomposition of the form

$$\mathbf{x}^* = \mathbf{y}^* + \lambda \mathbf{a}^*, \qquad \mathbf{y}^* \in \alpha^*. \tag{5.12}$$

For $\mathbf{y}^* = \mathbf{x}^* - \lambda \mathbf{a}^*$ lies in α^* if and only if $\mathbf{a}^{*\prime} J \mathbf{y}^* = \mathbf{a}^{*\prime} J \mathbf{x}^* - \lambda \mathbf{a}^{*\prime} J \mathbf{a}^* = 0$, *i.e.*, if

$$\lambda = \mathbf{a}^{*\prime} J \mathbf{x}^* / \|\mathbf{a}^*\|. \tag{5.13}$$

We wish to construct a linear transformation S^* of A_{n+2} into itself which keeps each point of α^* fixed and which maps the pseudo-normal vector $\mathbf{a}^*$ of α^* onto $-\mathbf{a}^*$. Thus

$$S^*(\mathbf{y}^* + \lambda \mathbf{a}^*) = \mathbf{y}^* - \lambda \mathbf{a}^* \tag{5.14}$$

and (5.12) and (5.13) imply

$$S^*(\mathbf{x}^*) = \mathbf{x}^* - 2\lambda \mathbf{a}^* = \mathbf{x}^* - 2[(\mathbf{a}^{*\prime} J \mathbf{x}^*)/\|\mathbf{a}^*\|] \cdot \mathbf{a}^*. \tag{5.15}$$

We find that our problem has exactly one solution, *viz.*, (5.15). It has the matrix

$$S^* = I_{n+2} - 2\mathbf{a}^* \mathbf{a}^{*\prime} J / \|\mathbf{a}^*\|. \tag{5.16}$$

By (5.14)

$$S^{*2} = I_{n+2}. \tag{5.17}$$

Let $\mathbf{y}^* \in \alpha^*$. Then

$$\begin{aligned}\|\mathbf{y}^* + \lambda\mathbf{a}^*\| &= (\mathbf{y}^{*\prime} + \lambda\mathbf{a}^{*\prime})J(\mathbf{y}^* + \lambda\mathbf{a}^*) \\ &= \mathbf{y}^{*\prime}J\mathbf{y}^* + \lambda\mathbf{a}^{*\prime}J\mathbf{y}^* + \lambda\mathbf{y}^{*\prime}J\mathbf{a}^* + \lambda^2\mathbf{a}^{*\prime}J\mathbf{a}^*.\end{aligned}$$

Since the middle terms both vanish, we obtain

$$\|\mathbf{y}^* + \lambda\mathbf{a}^*\| = \|\mathbf{y}^* - \lambda\mathbf{a}^*\| = \|\mathbf{y}^*\| + \lambda^2\|\mathbf{a}^*\|. \tag{5.18}$$

In particular, the point $\mathbf{x}^* = \mathbf{y}^* + \lambda\mathbf{a}^*$ and its image (5.14) have the same norm or

$$\|S^*\mathbf{x}^*\| = \|\mathbf{x}^*\| \tag{5.19}$$

for all $\mathbf{x}^*$. Hence S^* belongs to the pseudo-orthogonal group Ω^* of A_{n+2} and we have

$$S^{*\prime}JS^* = J. \tag{5.110}$$

Obviously, S^* is uniquely determined by α^*. We call S^* the pseudo-euclidean symmetry with respect to α^*.

5.2. Suppose that we are given two distinct vectors $\mathbf{b}^*$ and $\mathbf{c}^*$ in A_{n+2} and we require an $(n+1)$-space α^* through O such that the (pseudo-euclidean) symmetry S^* with respect to α^* maps $\mathbf{b}^*$ onto $\mathbf{c}^*$. Thus

$$\mathbf{c}^* = S^*\mathbf{b}^*. \tag{5.21}$$

On account of (5.19) we have to assume

$$\|\mathbf{b}^*\| = \|\mathbf{c}^*\|, \tag{5.22}$$

i.e., $\mathbf{b}^*$ and $\mathbf{c}^*$ have the same "pseudo-euclidean length." By (5.12) we may put

$$\mathbf{b}^* = \mathbf{y}^* + \mathbf{a}^*, \qquad \mathbf{c}^* = \mathbf{y}^* - \mathbf{a}^*, \tag{5.23}$$

where $\mathbf{y}^* \in \alpha^*$ and $\mathbf{a}^*$ is pseudo-normal to α^*. Hence

$$\mathbf{a}^* = \tfrac{1}{2}(\mathbf{b}^* - \mathbf{c}^*) \tag{5.24}$$

and there is not more than one $(n+1)$-space α^* which solves our problem, *viz.*, the one determined by $\mathbf{a}^*$. It now follows from **5.1** that α^* is a solution if and only if $\mathbf{b}^*$ and $\mathbf{c}^*$ satisfy not only (5.21) but also

$$\|\mathbf{a}^*\| = \tfrac{1}{4}\|\mathbf{b}^* - \mathbf{c}^*\| \neq 0. \tag{5.25}$$

By (5.22), the last condition can be given the form

$$\mathbf{b}^{*\prime}J\mathbf{c}^* \neq \|\mathbf{b}^*\|. \tag{5.26}$$

Suppose now that (5.22) is satisfied but not (5.26). Put

$$\delta = \|\mathbf{b}^*\| = \|\mathbf{c}^*\| = \mathbf{b}^{*\prime}J\mathbf{c}^*. \tag{5.27}$$

Thus no symmetry will now map $\mathbf{b}^*$ onto $\mathbf{c}^*$. Since a symmetry maps O onto itself, no product of symmetries will map a point $\neq O$ onto O. Assume therefore

$$\mathbf{b}^* \neq 0, \qquad \mathbf{c}^* \neq 0. \tag{5.28}$$

J being regular, the quadric $\|\mathbf{x}^*\| = \delta$ is either regular ($\delta \neq 0$) or once degenerate ($\delta = 0$). Thus, if $n \geqq 1$, it cannot be the union of two $(n+1)$-spaces and there is a point $\mathbf{d}^*$ such that

$$\|\mathbf{d}^*\| = \delta; \qquad \mathbf{b}^{*\prime} J \mathbf{d}^* \neq \delta, \qquad \mathbf{c}^{*\prime} J \mathbf{d}^* \neq \delta. \tag{5.29}$$

Hence from the above there are two symmetries S_1^* and S_2^* mapping $\mathbf{d}^*$ onto $\mathbf{b}^*$ respectively $\mathbf{c}^*$. This implies: assume (5.27) and (5.28); then there are two symmetries S_1^* and S_2^* such that

$$\mathbf{c}^* = S_2^* S_1^* \mathbf{b}^*. \tag{5.210}$$

5.3. Given a point $\mathbf{b}^* \in A_{n+2}$, $\mathbf{b}^* \neq 0$, and a transformation

$$T^* \in \Sigma^*, \qquad T^* \notin \Sigma_0^*. \tag{5.31}$$

Here $\Sigma_0^* = \Sigma_0^*(\mathbf{b}^*)$ denotes the subgroup of all the transformations of Σ^* with the fixed point $\mathbf{b}^*$. By (4.21), there is a $\rho \neq 0$ such that

$$\|T^* \mathbf{x}^*\| = \rho \cdot \|\mathbf{x}^*\| \quad \text{for all } \mathbf{x}^*. \tag{5.32}$$

We assume

$$\|T^* \mathbf{b}^*\| = \|\mathbf{b}^*\|. \tag{5.33}$$

Thus either $\rho = 1$, *i.e.*, $T^* \in \Omega^*$, or $\|\mathbf{b}^*\| = 0$.

If

$$\mathbf{b}^{*\prime} J T^* \mathbf{b}^* \neq \|\mathbf{b}^*\|, \tag{5.34}$$

5.2 implies that there is one and only one pseudo-euclidean symmetry S^* which interchanges $\mathbf{b}^*$ and $T^* \mathbf{b}^*$. Hence

$$V^* = S^* T^* \in \Sigma_0^*. \tag{5.35}$$

By (5.17), (5.35) is equivalent to

$$T^* = S^* V^*. \tag{5.36}$$

Suppose conversely that a decomposition (5.36) of T^* is given where S^* is a symmetry and $V^* \in \Sigma_0^*$. Then from (5.31)

$$\mathbf{c}^* = S^* \mathbf{b}^* = T^* \mathbf{b}^* \neq \mathbf{b}^*. \tag{5.37}$$

By **5.2**, S^* is uniquely determined and (5.35) then determines V^*. Furthermore, (5.37) and (5.22) imply (5.33), while (5.34) follows from (5.26). We thus have: Let T^* satisfy (5.31). Then there is not more than one decomposition (5.36) and such a decomposition exists if and only if (5.33) and (5.34) hold.

By (5.31), $T^{*-1} \notin \Sigma_0^*$. Further, (5.33) and (5.34) are equivalent to

$$\|T^{*-1}\mathbf{b}^*\| = \|\mathbf{b}^*\|$$

and

$$\mathbf{b}^{*\prime} J T^{*-1}\mathbf{b}^* \neq \|\mathbf{b}^*\|,$$

respectively. Thus we may apply the above result to T^{*-1} instead of T^* obtaining: If (5.31) holds, there is at most one decomposition

$$T^* = W^* S^*, \tag{5.38}$$

where $W^* \in \Sigma_0^*$ and S^* is a symmetry; a decomposition (5.38) exists if and only if (5.33) and (5.34) are satisfied.

We now replace (5.33) and (5.34) by

$$\|T^*\mathbf{b}^*\| = \|\mathbf{b}^*\| = \mathbf{b}^{*\prime} J T^* \mathbf{b}^*, \tag{5.39}$$

and assume $n \geqq 1$. Thus (5.27) and (5.28) hold for $\mathbf{c}^* = T^*\mathbf{b}^*$, and by **5.2** there are two symmetries S_1^* and S_2^* such that $T^*\mathbf{b}_2^* = S_2^* S_1^* \mathbf{b}^*$. Hence T^* permits a decomposition

$$T^* = S_2^* S_1^* V^*, \tag{5.310}$$

where

$$V^* = S_1^* S_2^* T^* \in \Sigma_0^*. \tag{5.311}$$

We thus obtain: Suppose $T^* \in \Sigma^*$ satisfies (5.33). Then T^* can be written as the product of a transformation in Σ_0^* and two or fewer pseudo-euclidean symmetries ($n \geqq 1$).

5.4. In the case $n = 0$ the last two paragraphs of **5.2** do not apply. The conic

$$K_1\colon \|\mathbf{x}^*\| = x_1^2 - x_2^2 = 0$$

splits into two straight lines through O and the points

$$\mathbf{b}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.41}$$

respectively.

Let $T^* \in \Sigma^*$. Then by (5.32)

$$\|T^*\mathbf{b}^*\| = \|T^*\mathbf{c}^*\| = 0. \tag{5.42}$$

Thus $T^*\mathbf{b}^*$ and $T^*\mathbf{c}^*$ lie in K_1. Since

$$\mathbf{b}^{*\prime} J \mathbf{c}^* = (1\ 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2, \tag{5.43}$$

(4.22) implies

$$(T^*\mathbf{b}^*)'J(T^*\mathbf{c}^*) = \rho\cdot\mathbf{b}^{*\prime}J\mathbf{c}^* = 2\rho \neq 0. \tag{5.44}$$

Hence there are two scalars $\beta\neq 0$ and $\gamma\neq 0$ such that either

$$T^*\mathbf{b}^* = \beta\mathbf{b}^*, \qquad T^*\mathbf{c}^* = \gamma\mathbf{c}^* \tag{5.45}$$

or

$$T^*\mathbf{b}^* = \beta\mathbf{c}^*, \qquad T^*\mathbf{c}^* = \gamma\mathbf{b}^*. \tag{5.46}$$

In either case, (5.44) implies

$$\beta\gamma = \rho. \tag{5.47}$$

Obviously, J is a pseudo-euclidean symmetry which interchanges $\binom{0}{1}$ and $-\binom{0}{1}$ and which keeps the points of the x_1-axis fixed. It interchanges $\mathbf{b}^*$ and $\mathbf{c}^*$. Thus a transformation which satisfies (5.45) can be written as the product of J with a transformation for which (5.46) holds.

Suppose in particular $T^*\in\Omega^*$, *i.e.*, $\rho=1$ and assume (5.46). Then (5.47) implies $T^*(\mathbf{b}^*+\beta\mathbf{c}^*)=\beta\mathbf{c}^*+\mathbf{b}^*$. Thus $\mathbf{d}^*=\mathbf{b}^*+\beta\mathbf{c}^*$ is invariant with respect to T^* while the vector $\mathbf{a}^*=\mathbf{b}^*-\beta\mathbf{c}^*$ is mapped onto $-\mathbf{a}^*$. As $\mathbf{a}^*$ is pseudo-normal to $\mathbf{d}^*$, T^* is simply the symmetry with respect to the line $\lambda\mathbf{d}^*$. We thus have: If $n=0$, then every transformation of Ω^* is either a pseudo-euclidean symmetry or the product of two such symmetries. In the latter case one of them may be chosen to be J. (Actually, it can be arbitrarily prescribed since any symmetry in A_2 must satisfy (5.46)).

5.5. We can now show that every transformation in Ω^* can be decomposed into a product of pseudo-euclidean symmetries.

By **5.4**, this statement is true for $n=0$. Thus we may assume that it is proved up to $n-1$. Denote the pseudo-orthogonal group of A_{n+2} by Ω_n^* and let $T_n^*\in\Omega_n^*$. Put $\mathbf{b}^{*\prime}=(1\ 0\ \cdots\ 0)$. If $\mathbf{b}^*$ is not a fixed point of T_n^*, T_n^* permits a decomposition (5.36) or (5.310) where $V^*\mathbf{b}^*=\mathbf{b}^*$. Since symmetries belong to Ω_n^*, (5.35) and (5.311) imply $V^*\in\Omega_n^*$. Thus T_n^* can always be expressed as a product of a transformation of Ω_n^* with the fixed point $\mathbf{b}_n^*$ and of symmetries. We may therefore assume

$$T_n^*\mathbf{b}^* = \mathbf{b}^*. \tag{5.52}$$

Put

$$J = J_n = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J_{n-1} \end{pmatrix} \qquad (n = 1, 2, \cdots). \tag{5.52}$$

By (4.23) $T_n^{*-1}=J_nT_n^{*\prime}J_n$, and T_n^{*-1} has the fixed point $\mathbf{b}^*$. Hence T_n^* has a matrix of the form

$$T_n^* = \begin{pmatrix} 1 & 0 \\ 0 & T_{n-1}^* \end{pmatrix} \tag{5.53}$$

and $T_n^{*\prime} J_n T_n^* = J_n$ implies

$$T_{n-1}^{*\prime} J_{n-1} T_{n-1}^* = J_{n-1}. \tag{5.54}$$

Thus T_{n-1}^* can be interpreted as a pseudo-euclidean transformation of the space

$$A_{n+1}\colon\ x_1 = \mathbf{b}^{*\prime} J \mathbf{x}^* = 0.$$

Let S_{n-1}^* be a symmetry in A_{n+1}. Thus S_{n-1}^* maps some vector $\mathbf{a}^* \in A_{n+1}$ onto $-\mathbf{a}^*$ and keeps the points of the n-space

$$\mathbf{a}^{*\prime} J_n \mathbf{x}^* = 0, \qquad x_1 = 0 \tag{5.55}$$

of A_{n+1} fixed. The symmetry

$$S_n^* = \begin{pmatrix} 1 & 0 \\ 0 & S_{n-1}^* \end{pmatrix} \tag{5.56}$$

of A_{n+2} will also map $\mathbf{a}^*$ onto $-\mathbf{a}^*$ and will keep the points of the $(n+1)$-space $\mathbf{a}^{*\prime} J_n \mathbf{x}^* = 0$ through the n-space (5.55) fixed. As $\mathbf{a}^* \in A_{n+1}$, this $(n+1)$-space passes through $\mathbf{b}^*$.

By the induction assumption, T_{n-1}^* can be decomposed into a product of symmetries S_{n-1}^* of A_{n+1}. Through (5.56), this determines a decomposition of T_n^* into symmetries S_n^* of A_{n+2}. This completes our proof.

6. Inversions I.

6.1. Again let

$$U = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -1 & 1 \\ 0 & 1 & 1 \end{array}\right); \tag{6.11}$$

cf. (3.32). Through

$$\mathbf{x} = U\mathbf{x}^*, \tag{6.12}$$

the points $\mathbf{x}^* \neq 0$ of A_{n+2} were mapped onto the spherical vectors $\mathbf{x}$ of complex conformal n-space C_n. Suppose in addition

$$\mathbf{a} = U\mathbf{a}^*. \tag{6.13}$$

Then the image of the $(n+1)$-space

$$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0 \tag{6.14}$$

is the linear pencil of the $(n-1)$-spheres $\mathbf{x}$ with

$$\mathbf{a}' G \mathbf{x} = 0; \tag{6.15}$$

cf. (3.33). It consists of the $(n-1)$-spheres orthogonal to $\mathbf{a}$. A point $\mathbf{x}$ satisfies (6.15) if and only if it is the center of a null sphere orthogonal to $\mathbf{a}$, *i.e.*, if and only if it lies on $\mathbf{a}$.

Through

$$T = UT^*U^{-1}, \tag{6.16}$$

the pseudo-euclidean similarity group Σ^* (the pseudo-orthogonal group Ω^*) of A_{n+2} was mapped onto the full conformal group Σ (the proper conformal group Ω) of C_n; *cf.* **4.3**.

6.2. A pseudo-euclidean symmetry S^* of A_{n+2} is mapped by U onto an *inversion*

$$S = US^*U^{-1} \tag{6.21}$$

of C_n. By (6.21) and (5.16)

$$\begin{aligned} S &= U(I_{n+2} - 2\mathbf{a}^*\mathbf{a}^{*\prime}J/\|\mathbf{a}^*\|)U^{-1} \\ &= I_{n+2} - 2U\mathbf{a}^*\cdot(U\mathbf{a}^*)'U^{-1}JU^{-1}/\|\mathbf{a}^*\|. \end{aligned}$$

Hence by (3.33), (3.38) and (6.13)

$$S = I_{n+2} - 2\mathbf{a}\mathbf{a}'G/\|\mathbf{a}\|. \tag{6.22}$$

By (5.11)

$$\|\mathbf{a}\| \neq 0 \tag{6.23}$$

and, by (5.17) and (6.21)

$$S^2 = I_{n+2}. \tag{6.24}$$

Since $S^* \in \Omega^*$, we have $S \in \Omega$; *cf.* **4.3**. Thus

$$S'GS = G \tag{6.25}$$

and

$$\|S\mathbf{x}\| = \|\mathbf{x}\|. \tag{6.26}$$

Formula (6.22) obviously implies

$$S\mathbf{a} = -\mathbf{a} \tag{6.27}$$

and

$$S\mathbf{x} = \mathbf{x} \tag{6.28}$$

for all $\mathbf{x}$ that satisfy (6.15). Thus S keeps each point of the $(n-1)$-sphere $\mathbf{a}$ fixed and maps every $(n-1)$-sphere orthogonal to $\mathbf{a}$ onto itself. Furthermore, (6.22) shows that $S\mathbf{b}$ is a multiple of $\mathbf{b}$ if and only if either $\mathbf{b}$ is a multiple of $\mathbf{a}$ or orthogonal to $\mathbf{a}$. Thus the above $(n-1)$-spheres are the only ones which are mapped by S onto themselves.

6.3. Let

$$\mathbf{a}'G\mathbf{b} \neq 0, \qquad \mathbf{c}'G\mathbf{b} = 0. \tag{6.31}$$

Then

$$\mathbf{c}'GS\mathbf{b} = \mathbf{c}'G\left(\mathbf{b} - 2\frac{\mathbf{a}'G\mathbf{b}}{\|\mathbf{a}\|}\cdot\mathbf{a}\right) = -2\frac{\mathbf{a}'G\mathbf{b}}{\|\mathbf{a}\|}\cdot\mathbf{c}'G\mathbf{a}.$$

Hence

$$\mathbf{c}'GS\mathbf{b} = 0 \leftrightarrow \mathbf{c}'G\mathbf{a} = 0. \tag{6.32}$$

We thus have: Suppose the $(n-1)$-sphere $\mathbf{b}$ is normal to $\mathbf{c}$ but not to $\mathbf{a}$. Then the $(n-1)$-sphere $\mathbf{c}$ is normal to $\mathbf{a}$ if and only if it is normal to $S\mathbf{b}$. In the special case that $\mathbf{b}=\mathbf{x}$ is a point we obtain: Suppose $\mathbf{x}$ does not lie on the $(n-1)$-sphere $\mathbf{a}$. Then an $(n-1)$-sphere through $\mathbf{x}$ is normal to $\mathbf{a}$ if and only if it passes through $S\mathbf{x}$.

We can now readily prove: If $\mathbf{x}$ and $\mathbf{y}$ are any two points, the four points

$$\mathbf{x}, \mathbf{y}, S\mathbf{x}, S\mathbf{y} \tag{6.33}$$

are concyclic, *i.e.*, they lie on a circle.

We may assume that the points (6.33) are mutually distinct, in particular,

$$S\mathbf{x} \neq \mathbf{x}, \qquad S\mathbf{y} \neq \mathbf{y}. \tag{6.34}$$

Let $\mathbf{c}$ be any $(n-1)$-sphere through the points

$$\mathbf{x}, \mathbf{y}, S\mathbf{x}. \tag{6.35}$$

Since $\mathbf{x}$ does not lie on $\mathbf{a}$, and since $\mathbf{x}$ and $S\mathbf{x}$ lie on $\mathbf{c}$, $\mathbf{c}$ is normal to $\mathbf{a}$. Since $\mathbf{y}$ lies on $\mathbf{c}$ but not on $\mathbf{a}$, $\mathbf{c}$ must therefore contain $S\mathbf{y}$. This holds for any choice of $\mathbf{c}$. Thus $S\mathbf{y}$ lies in the intersection of all the $(n-1)$-spheres through the points (6.35), *i.e.*, on the circle through them.

We note: If the points (6.33) are mutually distinct, the circle through them is normal to $\mathbf{a}$.

6.4. Let $\mathbf{n}$, where $\mathbf{n}' = (0 \cdots 0\ 1)$, denote the real absolute point of C_n; *cf.* (1.53). Let $\mathbf{a}$ be a proper $(n-1)$-sphere. By **2.1** this is equivalent to

$$a_{-1} = -2\mathbf{n}'G\mathbf{a} \neq 0. \tag{6.41}$$

By (3.52) the radius r of $\mathbf{a}$ satisfies

$$r^2 = \|\mathbf{a}\| \cdot a_{-1}^{-2}. \tag{6.42}$$

Suppose the inversion (6.22) maps $\mathbf{n}$ onto the point $\mathbf{m}$. Then

$$\mathbf{m} = S\mathbf{n} = \mathbf{n} - \frac{2\mathbf{a}'G\mathbf{n}}{\|\mathbf{a}\|}\cdot\mathbf{a} = \mathbf{n} + \frac{a_{-1}}{\|\mathbf{a}\|}\cdot\mathbf{a} = \mathbf{n} + \frac{\mathbf{a}}{a_{-1}r^2}$$

or

$$\mathbf{m}' = \{1/(a_{-1}r^2)\}(a_1, \cdots, a_n, a_{-1}, a_0/a_{-1}). \tag{6.43}$$

Thus $\mathbf{m}$ is the center of $\mathbf{a}$ with the weight r^{-2}.

Choosing $\mathbf{y}=\mathbf{n}$ in (6.33), we have: If $a_{-1}\neq 0$, the points

$$\mathbf{m},\ \mathbf{x},\ S\mathbf{x} \tag{6.44}$$

are collinear.

Now let $\mathbf{x}$ denote any finite point of C_n. Then (3.310) and (6.25) imply

$$\|S\mathbf{x}-\mathbf{m}\| = -2(S\mathbf{x})'G\mathbf{m} = -2\mathbf{x}'S'GS\mathbf{n} = -2\mathbf{x}'G\mathbf{n}$$

or

$$\|S\mathbf{x}-\mathbf{m}\| = x_{-1} = \text{weight of } \mathbf{x}. \tag{6.45}$$

Replacing $\mathbf{x}$ by $S\mathbf{x}$ and taking (6.24) into account, we obtain

$$\|\mathbf{x}-\mathbf{m}\| = \text{weight of } S\mathbf{x}. \tag{6.46}$$

Suppose in addition that $\mathbf{x}$ does not lie on the null sphere about $\mathbf{m}$. We apply (3.41) twice and make use of (6.46), (6.45), and (6.43) obtaining: The product of the squares of the distances of $\mathbf{x}$ and $S\mathbf{x}$ from $\mathbf{m}$ is equal to

$$\frac{\|\mathbf{x}-\mathbf{m}\|}{\text{weight of } \mathbf{x}\cdot\text{weight of } \mathbf{m}}\cdot\frac{\|S\mathbf{x}-\mathbf{m}\|}{\text{weight of } S\mathbf{x}\cdot\text{weight of } \mathbf{m}}$$

$$=\frac{\text{weight of } S\mathbf{x}}{\text{weight of } \mathbf{x}\cdot\text{weight of } \mathbf{m}}\cdot\frac{\text{weight of } \mathbf{x}}{\text{weight of } S\mathbf{x}\cdot\text{weight of } \mathbf{m}} = (\text{weight of } \mathbf{m})^{-2} = r^4.$$

This yields the well-known result that the product of the distances of $\mathbf{x}$ and $S\mathbf{x}$ from $\mathbf{m}$ is equal to $\pm r^2$.

6.5. The results of **6.4** can also be obtained more directly from (6.22). Put $\mathbf{y}=S\mathbf{x}$ and assume that $\mathbf{x}$ and $\mathbf{y}$ are both finite. Then

$$\frac{y_\lambda}{y_{-1}}-\frac{a_\lambda}{a_{-1}}=\frac{x_\lambda-2\mathbf{a}'G\mathbf{x}\cdot\|\mathbf{a}\|^{-1}\cdot a_\lambda}{x_{-1}-2\mathbf{a}'G\mathbf{x}\cdot\|\mathbf{a}\|^{-1}\cdot a_{-1}}-\frac{a_\lambda}{a_{-1}}$$

readily implies

$$\frac{y_\lambda}{y_{-1}}-\frac{a_\lambda}{a_{-1}}=\frac{x_{-1}}{y_{-1}}\left(\frac{x_\lambda}{x_{-1}}-\frac{a_\lambda}{a_{-1}}\right) \qquad (\lambda=1,\cdots,n). \tag{6.51}$$

We now introduce the inhomogeneous coordinate vectors ξ of $\mathbf{x}$, $\mathfrak{n}$ of $\mathbf{y}$, and $\mathfrak{y}$ of $\mathbf{m}$; *cf.* (1.36) and (2.15). Then by (6.51)

$$\mathfrak{n}-\mathfrak{y}=(x_{-1}/y_{-1})(\xi-\mathfrak{y}). \tag{6.52}$$

In particular the points (6.44) are collinear.

From (3.41) and (6.46) we can again deduce $(\xi-\mathfrak{y})^2=y_{-1}/(x_{-1}\cdot m_{-1})$ $=(y_{-1}/x_{-1})\cdot r^2$ or

$$x_{-1}/y_{-1}=r^2/(\xi-\mathfrak{y})^2. \tag{6.53}$$

Combining this formula with (6.52), we finally obtain

$$\mathfrak{n} - \mathfrak{y} = \frac{r^2}{(\xi - \mathfrak{y})^2} \cdot (\xi - \mathfrak{y}). \tag{6.54}$$

This formula implies in particular the last result of **6.4**, *viz.*,

$$(\mathfrak{n} - \mathfrak{y})^2, (\xi - \mathfrak{y})^2 = r^4. \tag{6.55}$$

We note that (6.52)–(6.55) break down if $\mathbf{x}$ lies on the null sphere about $\mathbf{m}$.

6.6. Let $\mathbf{a}$ be a nonisotropic $(n-1)$-space. Thus

$$a_{-1} = 0 \tag{6.61}$$

and

$$\|\mathbf{a}\| = \sum_1^n a_\lambda^2 \neq 0. \tag{6.62}$$

Let S denote the inversion with respect to $\mathbf{a}$ and put $\mathbf{y} = S\mathbf{x}$; *cf.* (6.22). Formula (1.36) determines the inhomogeneous coordinates of the points $\mathbf{x}$ and $\mathbf{y}$. The inhomogeneous equation of $\mathbf{a}$ is given by (2.11) and (6.61). Thus we can verify that S is the symmetry with respect to $\mathbf{a}$.

The following proof requires no computation: By (6.61) $y_{-1} = x_{-1}$. Hence S is a similarity; *cf.* (4.56). On account of (6.24), S must therefore be an orthogonality. Since S is not the identity and keeps the points of the $(n-1)$-space $\mathbf{a}$ fixed, our result follows.

6.7. By means of **6.1**, the properties of the pseudo-euclidean symmetries obtained in **5** can be translated into properties of inversions.

Belonging to Ω, inversions preserve norms. Let $\mathbf{b}$ and $\mathbf{c}$ be two $(n-1)$-spheres with the same norms. By **5.2** there is not more than one inversion mapping $\mathbf{b}$ onto $\mathbf{c}$. There is one if and only if

$$\mathbf{b}'G\mathbf{c} \neq \|\mathbf{b}\|. \tag{6.71}$$

If (6.71) does not hold, *i.e.*, if the $(n-1)$-spheres $\mathbf{b}$ and $\mathbf{b}-\mathbf{c}$ are orthogonal, $\mathbf{b}$ can be mapped onto $\mathbf{c}$ by a product of two inversions.

We now apply **5.3**. Let $T \in \Sigma$ and let $\mathbf{b}$ be fixed; $T\mathbf{b} \neq \mathbf{b}$. Then there is not more than one decomposition

$$T = SV \tag{6.72}$$

of T into an inversion S and a transformation $V \in \Sigma$ with

$$V\mathbf{b} = \mathbf{b}. \tag{6.73}$$

Such a decomposition exists if and only if

$$\|T\mathbf{b}\| = \|\mathbf{b}\| \tag{6.74}$$

and

$$\mathbf{b}'GT\mathbf{b} \neq \|\mathbf{b}\|. \tag{6.75}$$

Condition (6.74) is satisfied if and only if either $\mathbf{b}$ is a null sphere or if $T \in \Omega$. If (6.74) holds but not (6.75), T permits a decomposition

$$T = S_1 S_2 V, \tag{6.76}$$

where V again satisfies (6.73) while S_1 and S_2 are two inversions.

Similar results hold with respect to decompositions $T = VS$.

By **5.5** every element of Ω can be decomposed into a product of inversions. Thus Ω is identical with the group generated by the inversions.

7. Inversions II.

7.1. We now specialize $\mathbf{b} = \mathbf{n}$. By **6.4**, the inversion

$$S = I_{n+2} - 2\mathbf{a}\mathbf{a}'G/\|\mathbf{a}\| \tag{7.11}$$

maps $\mathbf{n}$ onto

$$S\mathbf{n} = \mathbf{n} + (a_{-1}/\|\mathbf{a}\|)\cdot\mathbf{a}. \tag{7.12}$$

Thus the following properties of S are equivalent:

(i) $S\mathbf{n} = \mathbf{n}$,
(ii) The point $S\mathbf{n}$ lies on the absolute null sphere,
(iii) $a_{-1} = 0$, *i.e.*, $\mathbf{a}$ is an $(n-1)$-space,
(iv) S is a symmetry,
(v) S is a similarity; *cf.* **4.5**.

Suppose the point $\mathbf{m}$ does not lie on the absolute null sphere. Then by **6.7**, there is one and only one inversion S mapping the spherical vector $\mathbf{n}$ onto the spherical vector $\mathbf{m}$. The spherical vector $\mathbf{a}$ defining S may be chosen to be

$$\mathbf{a} = \mathbf{m} - \mathbf{n}. \tag{7.13}$$

Thus

$$\|\mathbf{a}\| = a_{-1} = m_{-1} \tag{7.14}$$

and

$$S = I_{n+2} - 2(\mathbf{m} - \mathbf{n})(\mathbf{m}' - \mathbf{n}')G/m_{-1}. \tag{7.15}$$

The point $\mathbf{m} = S\mathbf{n}$ is the center of the $(n-1)$-sphere (7.13). By (6.42), the square of its radius is given by

$$r^2 = 1/m_{-1}. \tag{7.16}$$

If the point $\mathbf{m} \neq \mathbf{n}$ lies on the absolute null sphere, no such inversion exists. However, $\mathbf{n}$ can be mapped onto $\mathbf{m}$ by a product of two inversions.

7.2. By **4.5** a transformation $V \in \Sigma$ is a similarity if and only if

$$V\mathbf{n} = \lambda\mathbf{n} \tag{7.21}$$

is a multiple of $\mathbf{n}$.

Let $T \in \Sigma$ and let

$$T = SV \tag{7.22}$$

be a decomposition of T into an inversion S and a similarity V. Obviously S is a similarity if and only if T itself is one. In this case, S may be an arbitrary symmetry and V is defined through

$$V = ST. \tag{7.23}$$

From now on we exclude the case that T is a similarity. Thus S is not a symmetry and the point $S\mathbf{n}$ does not lie on the absolute null sphere. By (7.21),

$$T\mathbf{n} = SV\mathbf{n} = \lambda S\mathbf{n}. \tag{7.24}$$

Hence T permits a decomposition (7.22) only if the point

$$\mathbf{m} = T\mathbf{n} \tag{7.25}$$

is finite. By **6.7** this condition is also sufficient and there exists, in particular, one and only one decomposition (7.22) with

$$V\mathbf{n} = \mathbf{n}. \tag{7.26}$$

Since then

$$S\mathbf{n} = \mathbf{m}, \tag{7.27}$$

S will be the inversion (7.15). Thus the similarity V is determined by (7.23). We wish to compute the factors by which V multiplies weights and distances.

Since $T \in \Sigma$, there is a scalar $\rho \neq 0$ such that

$$T'GT = \rho G; \tag{7.28}$$

cf. (4.31). Hence by (7.23) and (6.25)

$$V'GV = T'S'GST = T'GT = \rho G; \tag{7.29}$$

therefore

$$\|V\mathbf{x}\| = \rho \cdot \|\mathbf{x}\|. \tag{7.210}$$

Let $\mathbf{z} = V\mathbf{x}$. Then (7.26) and (7.29) imply

$$z_{-1} = -2\mathbf{n}'G\mathbf{z} = -2\mathbf{n}'V'GV\mathbf{x} = -2\rho \cdot \mathbf{n}'G\mathbf{x}$$

or

$$z_{-1} = \rho x_{-1}. \tag{7.211}$$

Thus V multiplies every weight by ρ. Incidentally, (7.211) verifies directly that V is a similarity. Formulas (7.210) and (7.211) enable us to apply (3.41) obtaining: V multiplies the squares of the distances of any two finite points by $1/\rho$.

7.3. Let $\lambda \neq 0$ be arbitrary and suppose the point (7.25) is finite. Applying **7.2** to $\lambda^{-1}T$ instead of T, we obtain: There exists exactly one decomposition

$$\lambda^{-1}T = S_\lambda \cdot \lambda^{-1}V_\lambda, \qquad i.e., \; T = S_\lambda V_\lambda, \tag{7.31}$$

where S_λ is an inversion and $\lambda^{-1}V_\lambda$ is a similarity satisfying

$$\lambda^{-1}V_\lambda \mathbf{n} = \mathbf{n}, \qquad i.e., \; V_\lambda \mathbf{n} = \lambda \mathbf{n}. \tag{7.32}$$

Thus V_λ itself is a similarity. By (7.32), (7.31), and (7.25).

$$S_\lambda \mathbf{n} = \lambda^{-1} \cdot S_\lambda \lambda \mathbf{n} = \lambda^{-1} \cdot S_\lambda V_\lambda \mathbf{n} = \lambda^{-1} T \mathbf{n} = \lambda^{-1} \mathbf{m}. \tag{7.33}$$

It follows from **7.1** that S_λ is the inversion with respect to the $(n-1)$-sphere about the point $\mathbf{m}$ whose radius r_λ satisfies

$$r_\lambda^2 = 1/\lambda^{-1} m_{-1} = \lambda / m_{-1} = \lambda r^2. \tag{7.34}$$

Let $\mathbf{x}$ be any finite point that does not lie on the null sphere about $\mathbf{m}$. Put

$$\mathbf{y} = S\mathbf{x}, \qquad \mathbf{z} = S_\lambda \mathbf{y} = S_\lambda S \mathbf{x}, \tag{7.35}$$

and let the points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{m}$ have the inhomogeneous coordinate vectors $\xi, \mathfrak{n}, \zeta, \mathfrak{y}$, respectively. By (6.54)

$$\mathfrak{n} - \mathfrak{y} = \frac{r^2}{(\xi - \mathfrak{y})^2} \cdot (\xi - \mathfrak{y})$$

and therefore

$$\zeta - \mathfrak{y} = \frac{r_\lambda^2}{(\mathfrak{n} - \mathfrak{y})^2} \cdot (\mathfrak{n} - \mathfrak{y}) = \frac{r_\lambda^2}{r^2} (\xi - \mathfrak{y}).$$

Hence by (7.34)

$$\zeta - \mathfrak{y} = \lambda(\xi - \mathfrak{y}). \tag{7.36}$$

Being a similarity, $S_\lambda S$ multiplies every weight by the same factor. By (7.27) and (7.33)

$$S_\lambda S \mathbf{m} = S_\lambda \mathbf{n} = \lambda^{-1} \mathbf{m}. \tag{7.37}$$

The last two formulas completely describe $S_\lambda S$: It is a dilatation in the ratio $1{:}\lambda$ with the center $\mathbf{m}$. It multiplies every weight by λ^{-1}.

From (7.31) and (7.22)

$$V_\lambda = S_\lambda T = (S_\lambda S) V. \tag{7.38}$$

Thus V_λ differs from V only by the dilatation $S_\lambda S$.

The proof of (7.210) remains valid for V_λ. Thus V_λ also multiplies every norm by ρ. We note the following corollaries of (7.38) and of our discussions of V in **7.2** and of $S_\lambda S$: V_λ multiplies weights by $\lambda^{-1}\rho$ and the squares of the distances of any two finite points by λ^2/ρ. In particular, V_λ is an orthogonality if and only if

$$\lambda^2 = \rho. \tag{7.39}$$

This yields: The transformation T can be decomposed into a product of an orthogonality and an inversion in exactly two ways.

(We have $V_{-\lambda} = (S_{-\lambda}S_\lambda)V_\lambda$ and $S_{-\lambda}S_\lambda$ is the reflection with respect to the point $\mathbf{m}$. It multiplies every weight by -1).

7.4. If $T \in \Sigma$ is not a similarity and if the point (7.25) lies on the absolute null sphere, no decomposition (7.22) exists. However, T will admit decompositions

$$T = S_1 S_2 V, \tag{7.41}$$

where S_1 and S_2 are inversions and V is a similarity; *cf.* **6.7**.

THE BRAUER-RADEMACHER IDENTITY

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1. Introduction. Let n and r be integers, $r>0$, and let $\phi(r)$ and $\mu(r)$ denote, respectively, the arithmetical functions of Euler and Möbius. The following remarkable identity was stated by H. Rademacher and proved by A. Brauer [1]:

$$\phi(r) \sum_{d|r,\ (d,\ n)=1} \frac{d}{\phi(d)} \mu(r/d) = \mu(r) \sum_{d|(n,\ r)} d\mu(r/d), \tag{1}$$

the summation on the left being over the divisors of r prime to n, and the one on the right over the common divisors of n and r. This formula is noteworthy simply by virtue of its arithmetical elegance; however, in another form ([3], Lemma 1), it has been shown to have interesting applications as well ([3], Sec. 5).

In this note we give a new proof of (1), as an illustration of an arithmetical inversion principle whose proof is contained in another paper [4]. This principle may be stated as follows. Let $f(n, r)$ be a (complex-valued) *even* function of $n \pmod r$; that is, $f(n, r) = f((n, r), r)$ for all values of n [2]. Then, if $r = r_1 r_2$, where r_1 and r_2 are positive integers, it follows that

$$f(n, r) = \sum_{d|(n,\ r)} g(d, r/d) \rightleftarrows g(r_1, r_2) = \sum_{d|r_1} f(r_1/d, r)\mu(d). \tag{2}$$

We also mention that (1) can be proved by means of an identity of Landau

([5], p. 88, *1). In fact, a wide generalization of the Brauer-Rademacher relation is deduced in [4] on the basis of an extension of the Landau identity.

We make a final comment concerning the method of proof. Our objective is not merely to prove an identity, but rather to express a particular function (as defined by the left of (1)) in a particular form (as on the right of (2)). The above inversion principle shows that this is *possible*, that the result is *unique*, and indicates an *explicit* procedure for obtaining the desired result.

2. Lemmas on $\phi(r)$ and $\mu(r)$. We recall first the factorability properties, $\phi(r)=\phi(r_1)\phi(r_2)$, $\mu(r)=\mu(r_1)\mu(r_2)$, where $r=r_1r_2$, $(r_1, r_2)=1$ ([5], Ch. 16). In addition, we recall the familiar evaluation ([5], Th. 6–7),

$$\phi(r) = \sum_{d|r} d\mu(r/d). \tag{3}$$

The following simple lemmas will also be needed.

LEMMA 1. *If p is a prime and l is an integer $\geqq 2$, then $\phi(p^l)=p\phi(p^{l-1})$.*

Proof. By (3), $\phi(p^t)=p^{t-1}(p-1)$ if $t>0$, and the lemma follows immediately.

We define the *core* $\gamma(r)$ of r to be 1 if $r=1$, and to be the product of the distinct prime divisors of r if $r>1$. In addition, we shall say that r is *primitive* if r contains no square divisors >1, otherwise, nonprimitive. Note that $\mu(r)\neq 0$ or $=0$ according as r is primitive or nonprimitive.

LEMMA 2. *If r is primitive and if s is an integer such that $\gamma(s)\,|\,r$, then (provided $s>1$)*

$$\sum_{d|r} \frac{d\mu(d)}{\phi(ds)} = 0. \tag{4}$$

Proof. Denote the left side of (4) by $B(s, r)$. If $r=r_1r_2$, $((r_1, r_2)=1)$, $s=s_1s_2>1$, $\gamma(s_1)\,|\,r_1$, $\gamma(s_2)\,|\,r_2$, then the factorability properties of $\phi(r)$ and $\mu(r)$ imply that $B(s, r)=B(s_1, r_1)B(s_2, r_2)$. Since either $s_1>1$ or $s_2>1$, it suffices to prove (4) in case $r=p$ (p prime), $s=p^k$, $k>0$. But $B(p, p^k)=1/\phi(p^k)-p/\phi(p^{k+1})$, which, by Lemma 1, has the value 0.

LEMMA 3. *If r is primitive and d is a divisor of r, then $\mu(r/d)=\mu(r)\mu(d)$.*

Proof. By the definition and factorability of $\mu(r)$, it follows (for r primitive) that $\mu(r/d)=\mu(r/d)\mu^2(d)=\mu(r)\mu(d)$.

LEMMA 4.

$$\sum_{d|r,\ (r/d,\ n)=1} \mu(d) = \begin{cases} \mu(r), & \text{if } r\,|\,n, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where the summation is over divisors d of r whose conjugate divisors r/d are prime to n.

Proof. The case $n=1$ of (5) is the familiar property ([5], Th. 6-5) of $\mu(r)$,

$$\sum_{d|r} \mu(d) = \epsilon(r), \tag{5a}$$

where $\epsilon(r)$ is defined to be 1 if $r=1$ and 0 if $r>1$. We therefore have

$$\begin{aligned}\sum_{d|r,\ (r/d,\ n)=1} \mu(d) &= \sum_{d|r} \mu(d)\epsilon((r/d,\ n)) \\ &= \sum_{d|r} \mu(d) \sum_{D|(r/d,\ n)} \mu(D) = \sum_{D|(n,\ r)} \mu(D) \sum_{d|(r/D)} \mu(d) \\ &= \sum_{D|(n,\ r)} \mu(D)\epsilon(r/d),\end{aligned}$$

and the lemma results by definition of $\epsilon(r)$.

3. Proof of the Brauer-Rademacher identity (1). Denote the left member of (1) by $E(n, r)$. It is evident that $E(n, r)$ is even (mod r); hence we may apply the inversion formula (2) to obtain

$$E(n, r) = \sum_{d|(n,\ r)} g(d, r/d), \tag{6}$$

where

$$g(r_1, r_2) = \sum_{D|r_1} E(r_1/D, r)\mu(D) \qquad (r = r_1r_2). \tag{7}$$

Therefore, by definition of $E(n, r)$, we obtain from (7) and rearrangement,

$$g(r_1, r_2) = \phi(r) \sum_{d|r} \frac{d}{\phi(d)} \mu(r/d) \sum_{D|r_1,\ (r_1/D,\ d)=1} \mu(D). \tag{8}$$

By Lemma 4, the inner sum in (8) is zero unless $r_1 \mid d$, in which case it has the value $\mu(r_1)$. Placing $d=r_1e$, (8) becomes then

$$g(r_1, r_2) = r_1\phi(r)\mu(r_1) \sum_{e|r_2} \frac{e}{\phi(r_1e)} \mu(r_2/e). \tag{9}$$

In (9), the factor $\mu(r_2/e)$ is zero unless $(r_2/e) \mid \gamma(r_2)$; hence we may put $e=(r_2/\gamma(r_2))\delta$ to obtain

$$g(r_1, r_2) = \frac{r\phi(r)\mu(r_1)}{\mu(r_2)} \sum_{\delta|\gamma(r_2)} \frac{\delta\mu(\gamma(r_2)/\delta)}{\phi(r\delta/\gamma(r_2))}. \tag{10}$$

Now place $r/\gamma(r_2)=R_1R_2$, where $\gamma(R_2) \mid r_2$, $(R_1, r_2)=1$. In particular, note that $R_1 \mid r_1$. Then (10) becomes, by Lemma 3 and the factorability of $\phi(r)$,

$$g(r_1, r_2) = \frac{r\phi(r)\mu(r_1)\mu(\gamma(r_2))}{\gamma(r_2)\phi(R_1)} \sum_{\delta|\gamma(r_2)} \frac{\delta\mu(\delta)}{\phi(R_2\delta)}. \tag{11}$$

By Lemma 2, the sum in (11) is 0 unless $R_2=1$. But $R_2=1$ would imply that

$r_1r_2=r=R_1\gamma(r_2)$, so that $(r_1/R_1)r_2=\gamma(r_2)$. Hence, $g(r_1, r_2)=0$ unless $r_2=\gamma(r_2)$, $r_1=R_1$, and in particular, unless r_2 is primitive and $(r_1, r_2)=1$. In view of the factor $\mu(r_1)$ in (11), it follows then that $g(r_1, r_2)=0$, in case r is nonprimitive; restated,

$$g(r_1, r_2) = 0 \qquad \text{if } \mu(r) = 0. \tag{12}$$

Hence, in the remainder of the proof, we suppose (unless indicated otherwise) that r is *primitive*, so that $r_2=\gamma(r_2)$, $R_2=1$, $R_1=r_1$, $(r_1, r_2)=1$. Hence (11) becomes, since $\mu(r)$ and $\phi(r)$ are factorable,

$$\begin{aligned} g(r_1, r_2) &= r_1\phi(r_2)\mu(r) \sum_{\delta \mid r_2} \frac{\delta\mu(\delta)}{\phi(\delta)} \\ &= r_1\mu(r) \sum_{\delta \mid r_2} \delta\mu(\delta)\phi(r_2/\delta). \end{aligned}$$

Application of (3) and Lemma 3 yields

$$g(r_1, r_2) = r_1\mu(r_1) \sum_{\delta \mid r_2} \delta\mu(\delta) \sum_{D \mid (r_2/\delta)} D\mu(\delta D). \tag{13}$$

Placing $D\delta=E$ in (13), one obtains

$$\begin{aligned} g(r_1, r_2) &= r_1\mu(r_1) \sum_{\delta \mid r_2} \mu(\delta) \sum_{E \mid r_2,\ \delta \mid E} E\mu(E) \\ &= r_1\mu(r_1) \sum_{E \mid r_2} E\mu(E) \sum_{\delta \mid E} \mu(\delta). \end{aligned}$$

By (5a) it follows that

$$g(r_1, r_2) = r_1\mu(r_1) \qquad \text{if } \mu(r) \neq 0. \tag{14}$$

Formulas (12) and (14) can be combined into a single formula, valid for all r,

$$g(r_1, r_2) = \mu(r)r_1\mu(r_2). \tag{15}$$

Hence (1) results from (6) and (15).

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A CONSTRUCTION OF THE NULL SPACE OF A LINEAR TRANSFORMATION

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1. The theorem proved in this note (Sec. 2) provides a simple construction of the null space of a linear transformation of one finite-dimensional vector space into another (Sec. 3) and hence a method for finding all solutions of a system of linear equations (Sec. 4). All results apply if the underlying field is replaced by a division ring (Sec. 5). A comparison is made with known results, in order to provide a unifying point of view (Sec. 6). The details in Sections 4 and 5, while not essential for proofs, contribute to such a comparison of various methods of solution.

We assume as known the elementary results of linear algebra, as presented by Birkhoff and Mac Lane ([1], Ch. VII and VIII) and Jacobson ([3], Ch. I and II). We adopt the notation and terminology of [3] except as otherwise indicated.

By a *generalized row echelon matrix** we mean a matrix, such as the one below, whose elements belong to a field Δ and which has the property that each nonzero row precedes each zero row and that in each nonzero row after the first the number of zeros preceding the first nonzero element is greater than that number in the preceding row. The row rank of such a matrix is clearly the number of its nonzero rows.

$$\begin{pmatrix} 0 & 5 & 7 & 4 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Every matrix (α) can be transformed into a generalized row echelon matrix by a sequence S of elementary row operations ([1], pp. 172–174) or, equivalently, by multiplication on the left by the nonsingular matrix which is the product of the elementary matrices corresponding to the operations of S ([1], pp. 228–230). This does not change the row rank of (α).

2. THEOREM. *Let A be a linear transformation of rank r of an m-dimensional vector space $\mathfrak{R}_1$ into an n-dimensional vector space $\mathfrak{R}_2$ over a field Δ. Let $(e_1, \cdots, e_m)$ be an ordered basis for $\mathfrak{R}_1$, $(f_1, \cdots, f_n)$ an ordered basis for $\mathfrak{R}_2$, (α) the $m \times n$*

* *Cf.* the notions of (*row*) *echelon matrix* ([1], pp. 175–176), *row-reduced echelon matrix* ([1], pp. 173–174), and (*row*) *Hermite matrix* ([5], p. 91). From a generalized row echelon matrix one obtains a row echelon matrix by requiring that the first nonzero element of each nonzero row be 1, and a row-reduced echelon matrix by requiring in addition that in every column which contains such a first nonzero element 1 all other elements be zero. A square row-reduced echelon matrix is turned into a row Hermite matrix by permuting its rows so that all first nonzero elements 1 appear on the diagonal.

matrix of A relative to these bases, i.e.,

$$(\alpha) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix},$$

where

$$e_jA = \sum_{k=1}^{n} \alpha_{jk} f_k, \qquad j = 1, \cdots, m. \tag{1}$$

Then any sequence of elementary row operations which takes (α) into a generalized row echelon matrix takes the $m \times m$ identity matrix 1 into an $m \times m$ matrix (μ) such that the $m-r$ vectors

$$u_i = \sum_{j=1}^{m} \mu_{ij} e_j, \qquad i = r+1, \cdots, m, \tag{2}$$

form a basis for the null space of A.

Proof. Let S be a sequence of elementary row operations which transform (α) into a generalized row echelon matrix (γ). Let (μ) be the nonsingular $m \times m$ matrix which is the product of the elementary matrices corresponding to the operations of S. These operations transform the matrix 1 into (μ) $1 = (\mu)$; moreover,

$$(\mu)(\alpha) = (\gamma). \tag{3}$$

The number of nonzero rows of (γ) is the same as the row rank r of the matrix (α), and the last $m-r$ rows of (γ) are zero. Hence, if we define $m-r$ vectors u_i by (2), we have, using (1) and (3),

$$u_iA = \sum_{j,k} \mu_{ij}\alpha_{jk} f_k = \sum_{k} \gamma_{ik} f_k = 0, \qquad i = r+1, \cdots, m,$$

and the u_i belong to the null space $\mathfrak{N}$ of A. (The same conclusion can be drawn from the observation that the "row vector" $(\eta_1, \cdots, \eta_n)$ associated with $u_iA = \sum_k \eta_k f_k$ is equal to the matrix product $(\mu_{i1}, \cdots, \mu_{im})(\alpha)$, where $(\mu_{i1}, \cdots, \mu_{im})$ is the row vector associated with u_i, [3], p. 37). But $\mathfrak{N}$ has dimension $m-r$, and, since (μ) is nonsingular, the u_i are linearly independent. Hence the u_i form a basis for $\mathfrak{N}$.

3. To construct the null space of a given linear transformation A of $\mathfrak{R}_1$ into $\mathfrak{R}_2$, one writes down the matrix (α) of A relative to bases $(e_1, \cdots, e_m)$ of $\mathfrak{R}_1$, $(f, \cdots, f_n)$ of $\mathfrak{R}_2$, transforms (α) into a generalized row echelon matrix (γ) by elementary row operations, and simultaneously applies the same operations to the $m \times m$ identity matrix 1, obtaining an $m \times m$ matrix (μ). Then the vectors (2) are a basis for the null space of A. Of course, (α) and the bases may be given in advance, defining A.

Remark 1. This construction is a generalization of a well-known one of the inverse of A and of (α) ([1], pp. 230–231) in the special case in which (α) is nonsingular, $n=m=r$, $(\gamma)=(\mu)(\alpha)$ is in row-reduced echelon form, hence $(\gamma)=1$, and $(\mu)1=(\mu)$ is the matrix of the inverse of A relative to $(f_1, \cdots, f_n)$, $(e_1, \cdots, e_m)$, and is the inverse of (α). (See also the end of Sec. 4, and Sec. 6.)

Remark 2. In the above proof we have taken the dimension l of the null space of A as known. However, the proof can easily be modified to determine l. Indeed, if we do not know l, the proof still shows that $l \geqq m-r$. Also, if (μ) is the matrix, relative to $(e_1, \cdots, e_m)$, of any ordered basis of $\mathfrak{N}_1$ obtained by extending a basis of the null space of A, (μ) is nonsingular, and therefore the matrix $(\mu)(\alpha)$, which has l zero rows,* has the same rank as (α), whence $l \leqq m-r$. This shows that $l=m-r$.

4. The above construction is equivalent to finding all solutions $(\xi_1, \cdots, \xi_m)$ of the system of homogeneous linear equations

$$\begin{aligned} \xi_1\alpha_{11} + \cdots + \xi_m\alpha_{m1} &= 0, \\ \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad &\cdot \\ \xi_1\alpha_{1n} + \cdots + \xi_m\alpha_{mn} &= 0, \end{aligned} \tag{4}$$

or of the corresponding vector equation in the space of n-tuples over Δ,

$$\xi_1 a_1 + \cdots + \xi_m a_m = 0, \tag{5}$$

where the $a_j, j=1, \cdots, m$, are the "row vectors of (α)," $(\alpha_{j1}, \cdots, \alpha_{jn})$, associated with the vectors (1). (Indeed, it is clear from the proof in Sec. 2 that $(\xi_1, \cdots, \xi_m)$ is a solution of (4), hence of (5), if and only if $\sum_j \xi_j e_j$ belongs to the null space of the linear transformation A considered in Secs. 2, 3. See also [3], p. 48.) Now, as is well known, one solution of the inhomogeneous system

$$\xi_1\alpha_{1k} + \cdots + \xi_m\alpha_{mk} = \delta_k, \qquad k = 1, \cdots, n, \tag{4'}$$

or of the corresponding vector equation

$$\xi_1 a_1 + \cdots + \xi_m a_m = d, \tag{5'}$$

where the a_j are as above and d is $(\delta_1, \cdots, \delta_n)$, can be found by transforming the matrix (α) into (generalized) row echelon form or into row-reduced echelon form;† and all other solutions can be found if all solutions of (4) or (5) are known. Hence we can find all solutions of (4′) or (5′) by transforming (α) into (generalized) row echelon form or row-reduced echelon form and combining the known method for finding one solution of (4′) or (5′) with the construction of the null space of A given above.

(If (γ) is the generalized row echelon matrix in (3) and the vectors c_i

* See the remark on row vectors in the proof in Section 2.

† See [1], pp. 178–179. In the procedure described there, the row echelon form could obviously be replaced by the generalized row echelon form, as is done below.

$=(\gamma_{i1}, \cdots, \gamma_{in})$, $i=1, 2, \cdots, r$, are the first r, linearly independent, row vectors of (γ), one first finds, as in [1], pp. 178–179, the only possible solution $(\eta_1, \cdots, \eta_r)$ of the vector equation

$$\eta_1 c_1 + \cdots + \eta_r c_r = d, \tag{6}$$

if any, and then substitutes for the c_i from the equations

$$c_i = \sum_{j=1}^{m} \mu_{ij} a_j, \qquad i = 1, \cdots, r,$$

derived from (3), to find a solution $(\xi_1, \cdots, \xi_m)$ of (5′), where

$$\xi_j = \sum_{i=1}^{r} \eta_i \mu_{ij}, \qquad j = 1, \cdots, m;$$

i.e., $(\xi_1, \cdots, \xi_m)$ is $\sum_{i=1}^{r} \eta_i(\mu_{i1}, \cdots, \mu_{im})$. To this vector is added the row vector corresponding to an arbitrary element of the null space of A,

$$\sum_{i=r+1}^{m} \lambda_i(\mu_{i1}, \cdots, \mu_{im}), \qquad \lambda_{r+1}, \cdots, \lambda_m \in \Delta,$$

to obtain the general solution of (5′). In matrix form this solution is

$$(\eta_1, \cdots, \eta_r, \lambda_{r+1}, \cdots, \lambda_m)(\mu), \qquad \lambda_{r+1}, \cdots, \lambda_m \in \Delta. \tag{7}$$

Here $\eta_1, \cdots, \eta_r$ can be successively determined from the "triangular" set of r equations

$$\eta_1 \gamma_{1t_i} + \cdots + \eta_i \gamma_{it_i} = \delta_{t_i}, \qquad i = 1, \cdots, r,$$

derived from (6), where t_i is the column number such that γ_{it_i} is the first non-zero element in the ith row of (γ). If (γ) is in row-reduced echelon form, these equations reduce to $\eta_i = \delta_{t_i}$, $i=1, \cdots, r$. In the special case in which (α) is nonsingular, $n=m=r$, and (γ) is 1, one has $(\mu)=(\alpha)^{-1}$ and (7) amounts to solving the matrix form of the equations (4′) by matrix inversion.)

5. The above results also apply if Δ is a division ring instead of a field and $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are left vector spaces over Δ. One only has to require that all multiplications of a row by a scalar which appear in elementary row operations, should be multiplications on the left. (See [3], pp. 19–21 for a similar argument.) To obtain these results for right vector spaces $\mathfrak{R}_{1'}$ and $\mathfrak{R}_{2'}$, one assigns to the linear transformation A with

$$e_j A = \sum_{k=1}^{n} f_k \alpha_{kj}, \qquad j = 1, \cdots, m,$$

where $(e_1, \cdots, e_m)$ is a basis for $\mathfrak{R}_{1'}$ and $(f_1, \cdots, f_n)$ is a basis for $\mathfrak{R}_{2'}$, the $n \times m$ matrix

$$(\bar{\alpha}) = \begin{bmatrix} \bar{\alpha}_{11} & \cdots & \bar{\alpha}_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \bar{\alpha}_{n1} & \cdots & \bar{\alpha}_{nm} \end{bmatrix},$$

which has column vectors associated with the e_iA. (See [3], pp. 49–50.) One replaces, in an obvious way, elementary row operations by elementary column operations ([1], p. 233) involving multiplication by scalars on the right, row echelon forms by the corresponding column echelon forms, the $m \times m$ matrix (μ) by an $m \times m$ matrix $(\bar{\mu})$, and the second members of (2) by

$$\sum_{j=1}^{m} e_j \bar{\mu}_{ji}, \qquad i = r+1, \cdots, m,$$

where $(\bar{\mu})$ is the result of transforming 1 by a sequence of elementary column operations which takes $(\bar{\alpha})$ into a generalized column echelon matrix $(\bar{\gamma}) = (\bar{\alpha})(\bar{\mu})$. In Section 4, the first members of (4) and (4′), and of (5) and (5′), are replaced by the righthanded terms $\bar{\alpha}_{k1}\xi_1 + \cdots + \bar{\alpha}_{km}\xi_m$, $k = 1, \cdots, n$, and $\bar{a}_1\xi_1 + \cdots + \bar{a}_m\xi_m$, respectively, where the $\bar{a}_j$ are the column vectors of $(\bar{\alpha})$. Row vectors are replaced by column vectors, and (7) becomes

$$(\bar{\mu})(\bar{\eta}_1, \cdots, \bar{\eta}_r, \bar{\xi}_{r+1}, \cdots, \bar{\xi}_m)',$$

where the second factor is defined in the obvious way and the dash indicates transpose.

6. The above method of finding the null space of a linear transformation (of a right vector space into a right vector space, say) differs from another, known one (described, for right vector spaces, in equational form in [1], pp. 180–181, and in matrix form in [5], pp. 78–80, 93 and in [4]) by using only columns of matrices and by using only the generalized (column) echelon form as against the (row-) reduced echelon form or the (row) Hermite form. But it also requires operations both on the given matrix $(\bar{\alpha})$ and on the identity matrix, as against operations on $(\bar{\alpha})$ only. The difference is that between finding the null space of a linear transformation A and finding the space incident with the rank space of its transpose A^* ([3], p. 59). Correspondingly, the solution of an inhomogeneous vector equation or system of linear scalar equations in a right vector space, say, may be found either by column operations, using the method* sketched in Section 5, or else, if this is more appropriate for the numerical problem at hand, by row operations. (For the latter method see, for example, [1], pp. 44–47; [5], pp. 106–108, which has computations to which ours are analogous; [4], which contains similar computations, related to a notion of generalized inverse applicable to the infinite-dimensional case; and [2], pp. 93–94.)

* The general solution mentioned in Section 5 can also be obtained from [1], p. 236, Ex. 12, by taking P and D to be, respectively, the identity matrix and a matrix in one of the column echelon forms.

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A TOURNAMENT PROBLEM

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Four bridge players (or tennis players [1]) wanting to play a fair "round robin" tournament would have no difficulty at all in arranging a schedule. Using the letters A, B, C, D, to denote the players they would automatically play the three matches

A and B against C and D
A and C against B and D
A and D against B and C.

This schedule has the following features:

(a) Each player plays exactly *once with* each other player.
(b) Each player plays exactly *twice against* each other player.
(c) No one plays *against* himself.
(d) No one plays *with* himself.

It is also true that each team plays the only possible team of opponents. When there are more players, this additional feature (each team playing all possible opponents) requires a very long tournament. The question arises, "Can a schedule be arranged for v players so that at least features (a), (b), (c), (d) are satisfied?"

For $v=5$ a little effort produces a successful result such as

$$AB - CD, \quad AC - DE, \quad AD - BE, \quad AE - BC, \quad BD - CE,$$

and further effort shows that, aside from relettering of the players, no other solution is possible.

Before considering larger values of v it is helpful to deduce the number of matches necessary, *assuming* that a solution is possible. Our problem is, of course, similar to that of constructing a block design. In the usual notation for the parameters of a block design we let

v = number of players,
$\lambda = 3$ (number of matches involving a given pair),

$k = 4$ (four players to a match),
b = number of matches,
$r = v - 1$ (number of times each individual plays),

and have at once the familiar

$$b = \{v(v-1)\}/4. \tag{1}$$

This makes it evident that solutions cannot exist except possibly for

$$v = 4, 5, 8, 9, 12, 13, \cdots.$$

The first two cases were covered at the outset. Some additional solutions follow, and others may be found in [4].

$v = 8$

AB – CD	BC – EH	
AC – FH	BD – EG	
AD – FG	BF – DH	
AE – CG	BG – CF	
AF – BE	CE – DF	
AG – BH	CH – DG	
AH – DE	EF – GH	

$v = 9$

AB – CD	AH – CF	BI – EH
AC – EI	AI – FG	CG – DF
AD – HI	BC – FI	CH – DE
AE – DG	BD – GI	CI – GH
AF – BE	BF – DH	EF – DI
AG – BH	BG – CE	EG – FH

$v = 12$*

AB – HI	AL – EH	AK – DG
EF – BG	CE – LF	KI – EJ
GH – DK	AE – IC	HK – EL
BI – CG	CD – EI	AF – JD
AC – KJ	FH – BK	DE – FJ
FG – CH	DF – IG	GK – CL
LJ – FB	AD – BL	AI – FK
CJ – DH	BD – KE	BC – DL
LI – GJ	BH – IJ	HL – DI
CK – BJ	EG – HJ	KL – FI
AG – BE	AJ – GL	AH – CF

$v = 13$

BM – IF	CL – DK	EJ – GH
CA – JG	DM – EL	FK – HI
DB – KH	EA – FM	GL – IJ
EC – LI	FB – GA	HM – JK
FD – MJ	GC – HB	IA – KL
GE – AK	HD – IC	JB – LM
HF – BL	IE – JD	KC – MA
IG – CM	JF – KE	LD – AB
JH – DA	KG – LF	ME – BC
KI – EB	LH – MG	AF – CD
LJ – FC	MI – AH	BG – DE
MK – GD	AJ – BI	CH – EF
AL – HE	BK – CJ	DI – FG

$v = 16$*

AB – GN	AP – CH	AC – HP	BC – MJ	AH – CP	GH – OE
BO – LI	MP – GI	CF – EI	AE – LJ	PJ – KO	MN – EK
CD – NO	GL – CK	LN – FP	AD – KI	NJ – HI	CJ – BM
KM – EN	AG – BN	FG – DJ	EH – GO	IJ – HN	BF – HK
IO – BL	DL – HM	AM – FO	BD – PE	AO – FM	DH – LM
KL – CG	AF – MO	KN – EM	DF – JG	LP – FN	PI – GM
AN – BG	DM – HL	OC – DN	EG – OH	AK – ID	NI – HJ
DO – CN	JO – KP	NP – FL	AL – JE	DE – BP	GK – CL
BE – DP	BJ – CM	FH – BK	AI – DK	AJ – EL	MI – GP
CE – IF	PO – KJ	DG – FJ	FK – BH	EF – IC	BI – LO

* The results for $v = 12$ and $v = 16$ were obtained by D. La Dage, Boston University.

An obvious generalization of our problem involves teams of m players. With the same notation as before and $k=2m$ we replace (a) and (b) by

(a^1) Each player plays exactly $m-1$ times with each other player.

(b^1) Each player plays exactly m times against each other player.

This implies $\lambda=2m-1$ and leads at once to

$$b = \{v(v-1)\}/(2m) \tag{2}$$

Again solutions are possible only for certain sequences. The following examples were obtained by manual rearrangement of standard block designs [3].

$m=3, v=7$	$m=3, v=9$	$m=4, v=8$
ABC – DEF	ABC – DEF	ABCD – EFGH
ADE – BFG	ABE – DGH	ABEF – CDGH
AFG – BCE	ADG – CFI	ABGH – CDEF
BDG – ACF	AFH – BDI	ACEG – BDFH
BEF – CDG	AEH – CFG	ACFH – BDEG
CDF – AEG	AFI – BEG	ADEH – BCFG
CEG – ABD	ACD – EHI	ADFG – BCEH
	AGI – BCH	
	BFH – CEI	
	BDI – CEG	
	BFG – CDH	
	DEF – GHI	

It is interesting to note that the first of these is, as printed, a pair of Steiner triple systems of order seven. The second, however, cannot be so divided. Perhaps some valiant or fortunate reader will discover a design for $m=3$, $v=10$ or $m=4$, $v=9$. A few moments is sufficient, however, to try the various combinations for $m=3$, $v=6$ and discover that no solution is possible even though (2), for these values, suggests $b=5$.

This impossibility raises the general question of existence. No effort has been made to discover the answer. However, R. C. Bose has suggested that his method of Galois fields [2] can be modified to cover certain sequences; indeed, the design listed earlier for $v=13$ was found by his method. Should the military forces of any friendly nation develop a lively interest in tournaments it is very likely that many hands will become available for the task. To further these prospects this brief paper should perhaps have been titled "On the testing of individual ability in group competition."

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A GENERALIZATION OF HERMITE'S INTERPOLATION FORMULA

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1. Introduction. Hermite's interpolation formula provides an expression for a polynomial which passes through given points with given slopes. Specifically, let x_i, f_i, $f_i^{(1)}$, $i=0, 1, \cdots, n$, be given. The polynomial $f(x)$, of degree $2n+1$, for which $f(x_i)=f_i$, $f'(x_i)=f_i^{(1)}$ is given by ([2], p. 33)

$$f(x) = \sum_{j=0}^{n} h_j(x)f_j + \sum_{j=0}^{n} \bar{h}_j(x)f_j^{(1)},$$

where

$$h_j(x) = \left\{1 - \frac{q_n''(x_j)}{q_n'(x_j)}(x - x_j)\right\}\{L_j(x)\}^2, \qquad \bar{h}_j(x) = (x - x_j)\{L_j(x)\}^2,$$

$$q_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n), \qquad L_j(x) = \frac{q_n(x)}{(x - x_j)q_n'(x_j)}.$$

It is the purpose of this paper to generalize Hermite's formula so as to include the values of derivatives up to specified, arbitrary orders at the given points.

2. The generalized interpolation formula. We state here the general result which is to be proved.

THEOREM 1. *Let there be given x_j, r_j, $f_j^{(k)}$, $j=0, 1, \cdots, n$; $k=0, 1, \cdots, r_j$. Let $p_j(x)$ and $g_j(x)$ be defined by*

$$(1)\quad p_j(x) = (x - x_0)^{r_0+1}\cdots(x - x_{j-1})^{r_{j-1}+1}(x - x_{j+1})^{r_{j+1}+1}\cdots(x - x_n)^{r_n+1},$$

$$g_j(x) = [p_j(x)]^{-1}.$$

Then the polynomial $f(x)$ of degree $n+\sum_{j=0}^{n} r_j$ such that

$$f^{(k)}(x_j) = f_j^{(k)}, \qquad j = 0, 1, \cdots, n, \; k = 0, 1, \cdots, r_j,$$

is given by

$$(2)\qquad f(x) = \sum_{j=0}^{n}\sum_{k=0}^{r_j} A_{jk}(x)f_j^{(k)},$$

where

$$(3)\qquad A_{jk}(x) = p_j(x)\frac{(x - x_j)^k}{k!}\sum_{t=0}^{r_j-k}\frac{1}{t!}g_j^{(t)}(x_j)(x - x_j)^t.$$

Before giving the proof let us see how this applies in the specific case covered by Hermite's formula. Here we have $r_j=1$ for $j=0, 1, \cdots, n$. We find

$$p_j(x) = (x - x_0)^2\cdots(x - x_{j-1})^2(x - x_{j+1})^2\cdots(x - x_n)^2.$$

Then for $k=1$ we have $A_{j1}(x)=p_j(x)(x-x_j)g_j(x_j)$. But $g_j(x_j)=[p_j(x_j)]^{-1}$ so that

$$A_{j1}(x) = (x - x_j)\frac{p_j(x)}{p_j(x_j)} = (x - x_j)\{L_j(x)\}^2,$$

which is the $\bar{h}_j(x)$ of Section 1. For $k=0$ we have

$$\begin{aligned} A_{j0}(x) &= p_j(x)[g_j(x_j) + g_j'(x_j)(x - x_j)] \\ &= p_j(x)\left[\frac{1}{p_j(x_j)} - \frac{p_j'(x_j)}{[p_j(x_j)]^2}(x - x_j)\right] \\ &= \left[1 - \frac{p_j'(x_j)}{p_j(x_j)}(x - x_j)\right]\{L_j(x)\}^2. \end{aligned}$$

But $p_j(x_j)=[q_n'(x_j)]^2$ and $p_j'(x_j)=q_n'(x_j)q_n''(x_j)$, so that $A_{j0}(x)$ turns out to be the $h_j(x)$ of Section 1, and Hermite's interpolation formula is seen to be a special case of the general formula.

3. Proof of the theorem. We seek a representation for $f(x)$ of the form (2). We must then have

$$A_{jk}^{(s)}(x_i) = 0, \qquad i \neq j, \qquad A_{jk}^{(s)}(x_j) = \delta_{ks} \tag{4}$$

where δ_{ks} is the Kronecker delta, the superscript (s) indicates order of derivative, and each $A_{jk}(x)$ is a polynomial of degree $n+\sum_{j=0}^{n} r_j$. It follows from (4) that

$$A_{jk}(x) = p_j(x)(x - x_j)^k R_{jk}(x), \tag{5}$$

where $R_{jk}(x)$ is a polynomial of degree r_j-k. Let this equation be rewritten as

$$S_{jk}(x)R_{jk}(x) = A_{jk}(x)g_j(x), \tag{6}$$

where $S_{jk}(x)=(x-x_j)^k$, $g_j(x)=[p_j(x)]^{-1}$. We now differentiate (6) $k+t$ times, keeping in mind that

$$S_{jk}^{(k+t)}(x) = \begin{cases} k! \text{ for } t = 0, \\ 0 \text{ for } t > 0. \end{cases}$$

We obtain

$$\sum_{i=0}^{k}\binom{k+t}{i}S_{jk}^{(i)}(x)R_{jk}^{(k+t-i)}(x) = \sum_{i=0}^{k+t}\binom{k+t}{i}A_{jk}^{(i)}(x)g_j^{(k+t-i)}(x). \tag{7}$$

Now let $x=x_j$ in (7). Since $S_{jk}^{(s)}(x_j)=0$ for $s=0, 1, \cdots, k-1$, and also using (4), we get

$$\binom{k+t}{t}k!R_{jk}^{(t)}(x_j) = \binom{k+t}{k}g_j^{(t)}(x_j)$$

so that

$$R_{jk}^{(i)}(x_j) = \frac{1}{k!}\, g_j^{(i)}(x_j), \qquad i \leqq r_j - k.$$

We recall that $R_{jk}(x)$ is a polynomial of degree $r_j - k$. Hence

$$R_{jk}(x) = \frac{1}{k!} \sum_{i=0}^{r_j-k} \frac{1}{i!}\, g_j^{(i)}(x_j)(x - x_j)^i.$$

With this substituted in (5) the $A_{jk}(x)$ are as in the statement of the theorem, and the proof is complete.

4. Divided differences with repeated arguments. We shall give an application of the theorem. Let

$$\left[\underbrace{x_0 \cdots x_0}_{r_0+1}\ \underbrace{x_1 \cdots x_1}_{r_1+1} \cdots \underbrace{x_n \cdots x_n}_{r_n+1}\right], \qquad \text{with } x_i \neq x_j \text{ for } i \neq j, \tag{8}$$

be the general divided difference with repeated arguments of a function $f(x)$. In ([3], p. 14) the value of this divided difference is given as

$$\frac{1}{r_0!r_1! \cdots r_n!}\, \frac{\partial^{r_0+r_1+\cdots+r_n}}{\partial x_0^{r_0}\, \partial x_1^{r_1} \cdots \partial x_n^{r_n}} [x_0x_1 \cdots x_n],$$

where

$$[x_0x_1 \cdots x_n] = \sum_{k=0}^{n} \left[f(x_k) \Big/ \prod_{\substack{i=0 \\ i\neq k}}^{n} (x_k - x_i) \right].$$

This divided difference may also be represented by the contour integral

$$\frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t - x_0)^{r_0+1} \cdots (t - x_n)^{r_n+1}},$$

where C is a simple closed curve enclosing a simply connected region of the complex variable t which contains the points $x_0, x_1, \cdots, x_n$, provided $f(t)$ is analytic within and on C.

We may derive yet another expression for the divided difference (8).

THEOREM 2. *The divided difference* (8) *is given by*

$$\sum_{j=0}^{n} \sum_{k=0}^{r_j} \frac{1}{k!}\, \frac{1}{(r_j - k)!}\, g_j^{(r_j-k)}(x_j) f_j^{(k)},$$

where $g_j(x)$ *is given in Theorem* 1.

Proof. The divided difference in question is the coefficient of the highest degree term in the generalized interpolation formula of Section 2. From (2) and (3) this coefficient is easily seen to be given by the above double sum.

COROLLARY. *If $n=0$ we take $p_0(x)\equiv 1$, and the above theorem gives the familiar result*

$$\left[\underbrace{x_0 \cdots x_0}_{r_0+1}\right] = \frac{1}{r_0!} f_0^{(r_0)}.$$

5. Evaluation of the confluent alternant. If we consider the system of equations for the general interpolation problem of Theorem 1, we have a set of equations consisting of the r_0+1 equations

$$\begin{aligned} a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + \cdots &= f_0^{(0)}, \\ a_1 + 2a_2x_0 + 3a_3x_0^2 + \cdots &= f_0^{(1)}, \\ 2a_2 + 6a_3x_0 + \cdots &= f_0^{(2)}, \end{aligned}$$

and so on, followed by similar sets of $r_1+1, r_2+1, \cdots, r_n+1$ equations for each of the values $x_1, x_2, \cdots, x_n$, respectively, with $N+1\equiv n+1+\sum_{i=0}^{n} r_i$ equations in all.

The determinant of this system, which we shall denote by $\Delta_{r_n}^n$, is the confluent alternant as described in Aitken ([5], p. 119), multiplied by

$$\prod_{i=0}^{n} [r_i!(r_i-1)!\cdots 2!1!].$$

Now $\Delta_{r_n}^n$ is the denominator of the coefficient of x^N, as determined from a system of equations such as above by Cramer's rule, for the polynomial of degree N such that

$$f(x_0) = f'(x_0) = \cdots = f^{(r_0)}(x_0) = f(x_1) = \cdots = f^{(r_1)}(x_1) = \cdots = f^{(r_n-1)}(x_n) = 0,$$
$$f^{(r_n)}(x_n) = 1.$$

In view of Theorem 1, this coefficient is also the coefficient of x^N in

$$\begin{aligned} A_{n,r_n} &= p_n(x)\cdot\frac{(x-x_n)^{r_n}}{r_n!}\cdot g_n(x_n) \\ &= g_n(x_n)\cdot\frac{(x-x_n)^{r_n}}{r_n!}\cdot(x-x_0)^{r_0+1}\cdots(x-x_{n-1})^{r_{n-1}+1} \end{aligned}$$

and is therefore

$$\frac{1}{r_n!}\,g_n(x_n) = \frac{1}{r_n!p_n(x_n)} = \frac{1}{r_n!(x_n-x_0)^{r_0+1}\cdots(x_n-x_{n-1})^{r_{n-1}+1}}.$$

The numerator of this coefficient is the determinant formed from $\Delta_{r_n}^n$ by deleting the last row and column, hence is similar to $\Delta_{r_n}^n$, and may be denoted by $\Delta_{r_n-1}^n$. We then have

$$\Delta^{n}_{r_n} = \Delta^{n}_{r_n-1} \cdot r_n!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1}.$$

In the same manner we obtain

$$\Delta^{n}_{r_n-1} = \Delta^{n}_{r_n-2} \cdot (r_n - 1)!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\Delta^{n}_{1} = \Delta^{n}_{0} \cdot 1!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

and also

$$\Delta^{n}_{0} = \Delta^{n-1}_{r_{n-1}}(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

where the determinant $\Delta^{n-1}_{r_{n-1}}$ is similar to $\Delta^{n}_{r_n}$, but is formed with the points $x_0, x_1, \cdots, x_{n-1}$ and is of order $n + \sum_{i=0}^{n-1} r_i$. Combining the above, we have

$$(9) \quad \Delta^{n}_{r_n} = \Delta^{n-1}_{r_{n-1}} \cdot r_n!(r_n - 1)! \cdots 2!1!\left[(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1}\right]^{r_n+1}.$$

Similar relations then exist for $\Delta^{i}_{r_i}$, $i = n-1, \cdots, 1, 0$, with

$$(10) \qquad \Delta^{0}_{r_0} = r_0!(r_0 - 1)! \cdots 2!1!.$$

A combination of relations of the type (9) for $\Delta^{i}_{r_i}$, $i = 0, 1, \cdots, n$, together with (10), gives the result

$$(11) \qquad \Delta^{n}_{r_n} = \prod_{i=0}^{n} [r_i!(r_i - 1)! \cdots 2!1!] \cdot \prod_{i>j} (x_i - x_j)^{(r_i+1)(r_j+1)}.$$

The value of the confluent alternant itself is therefore the product $\prod_{i>j} (x_i - x_j)^{(r_i+1)(r_j+1)}$ which is the value given by Aitken.

In the case for which all the r_i are equal, $r_i \equiv r$, $i = 0, 1, \cdots, n$, we may denote the determinant by Δ_r, for which (11) reduces to

$$\Delta_r = [r!(r - 1)! \cdots 2!1!]^{n+1} \cdot V^{(r+1)^2},$$

where V is the Vandermonde determinant whose element in the ith row and jth column is x_{i-1}^{j-1}, $i, j = 1, \cdots, n+1$.

6. Conclusion. The main result of this paper concerns an explicit representation of the interpolation polynomial. A discussion of the remainder term may be found in [1] and [4], in both of which an interpolation is obtained in a form which involves an iterative process.

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ON THE DESIRABILITY OF PUBLISHING CLASSIFIED BIBLIOGRAPHIES OF THE MATHEMATICS LITERATURE*

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Having been engaged now in mathematical research for 60 years, I find that certain crying needs stand out very clearly as to the things required by research mathematicians in order to produce and publish more articles, particularly original ones. To my mind, first among these needs is the preparation and publication of bibliographies classified according to topics of all branches in pure and applied mathematics. Accordingly, this paper will be devoted to this topic and will be divided into two parts. The first part will consist of a discussion of a bibliography of the Theory of Numbers either with reports, as a continuation of Dickson's *History of the Theory of Numbers*, cr a preparation of a bibliography consisting of titles only without reports, covering the literature since the year 1923. The second part will consist mainly of consideration of the advisability of publishing bibliographies of the various parts of mathematics without reviews.

First, we shall illustrate the kind of thing which mars a number of papers in number theory that appear at the present time, and probably we might note a great many more marred in a similar way if we knew more about the history of the topic being discussed. Some years ago two great mathematicians (whom we shall call X and Y) published an article. In their introduction they summarized, without reference, two results as being the principal ones being proved in their paper. One of these results was published without proof by a mathematician some 90 years before. The other result was stated and proved in full some 20 years before, by another mathematician. If I had written X and Y, or published a paper describing their derelictions, they could have replied that a paper of mine, which I published in 1944, contained a result and proof which was 109 years old. How can such duplications be avoided?

A number of mathematicians have expressed the hope that Dickson's *History* be brought up to date. Before doing so, it might be well to examine in detail some of the salient features of the book with a view to preserving them in the proposed extension. As far as I am concerned, I cannot do better than to quote something by Dickson himself (preface to Volume II):

> Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history.

* The work on this paper was done under National Science Foundation Grant 3697.

I shall also quote several excerpts from a review I wrote of Dickson's Vols. I and II (*Bull. Amer. Math. Soc.*, vol. 30, 1924, pp. 65–70):

> In the reviewer's opinion, Dickson has come very close to realizing the ideals above mentioned, and this circumstance gives the work its greatest value. In particular, it appears that he has had always in mind the needs of the investigator. An endeavor is made to supply every scrap of information that might possibly aid in research on a particular problem. . . .
>
> The material shows in itself how useful the history will be in enabling investigators to avoid duplication in published results. It is to be noted in glancing over the pages how often the contents of a published article is, unknown to the writer of it, largely a repetition of previous research of some other author. . . .
>
> It often happens in the history of mathematics that a mathematician becomes a specialist in a particular topic, and, after years of experience with it, he publishes a treatise giving a harmonious and comprehensive development of the subject, the material being all arranged and presented according to his own particular point of view. This treatise may become a classic, and its readers are likely to get in the habit of ignoring, to a considerable extent, the literature that preceded its publication. In this way the points of view of the older writers are often lost sight of, as these treatises rarely, if ever, reproduce all the older material on a particular topic. It would seem that there is too great a preponderance of books of this sort in the literature and too few histories or reports of the type of Dickson's work.

In view of the characteristics of the history just discussed, we see that it is easy to verify that it is as useful now (as far as it went) as it was in 1924, the year of our review. Therefore we should try to add to his work without destroying the general plan of the history. First, the magnitude of this task must be recognized.

We have made an estimate of the number of references that it would be necessary to review between 1920 and 1956 inclusive; it is 8,500. To illustrate what that means, suppose it was possible to persuade five mature, A-1 number theorists to undertake this job, and let us assume that they had five years to complete it. This would mean that if the references were evenly distributed, each person would have to dispose of 1,640 references in the five years, which would mean that 328 reviews would have to be completed in each year by each person. That is almost one reference per day. Considering the extremely advanced and complicated papers which have been written in the interval mentioned on analytical number theory and algebraic number theory, I do not regard any number theorist I know of as being capable of carrying out his part in such a work within the time specified.

To illustrate further the magnitude of the above-mentioned task, we counted the number of reviews that Dickson made in his three volumes. The total turned out to be 8,382, and this would not include references on algebraic numbers. Also, he did not include Bernoulli and related numbers and the law of quadratic reciprocity. On the average, the papers he reviewed were far less difficult and complicated than those we are considering.

Another point in connection with this is that few, if any, number theorists would wish to work on such a project full time for perhaps more than a year at a

time, since most research mathematicians are university professors. However, they would gain the advantage of a detailed knowledge of the literature.

In view of the above remarks, it might be possible for ten collaborators working more or less in a period of ten years to bring things nearly up to date. When I say "nearly" I have in mind the fact that *during the ten years just mentioned probably at least 4,000 more papers would be published.*

Now we shall consider the advisability of publishing bibliographies of the various topics in mathematics without reviews. I think it should be obvious that we cannot hope ever to have completed and published a bibliography of any extensive branch of mathematics (*with reports* in the style of Dickson's *History*) other than in number theory. To illustrate this we will consider the subject of differential equations. To write the history (à la Dickson) of this from its beginning, would be impossible, in my opinion, in any form at this late stage, and this is more or less true for other topics not number theoretic. I think, however, that the *preparation of a nearly complete list of references might be possible.* (I use the word *nearly* here since we can hardly hope to ever locate all the existing ones.) Of course if we list references in a particular topic we should be careful, if necessary, to divide the literature on the topic into sub-topics so that a research man does not have to look up too many listed papers to decide that results he has arrived at are new. For example, we could hardly expect him to examine 1,000 references. In an issue of *Mathematical Reviews* the subject index for the year 1955 listed some 120 papers on "ordinary differential equations: existence and behavior of solutions." So this topic should be separated into quite a number of sub-topics, if humanly possible.

Now I do not expect that the collecting of a large number of references on any mathematical topic and preparing a bibliography consisting of titles only will appeal to many mathematicians. The work of a bibliophile on first glance is not attractive. Often when I have mentioned the importance of such work to a research mathematician he has registered his deep disgust at the idea of his possibly being involved in it. *But here I do not think he realizes the benefits which will probably accrue to him personally and to the science of mathematics through work of this kind. I have already mentioned the advantageous effect on a mathematician who collects references and writes reports on each. To a considerable extent this is also true when one only examines a reference and decides what topics have been treated among the separate topics initially chosen. This process requires one to read rather carefully some of the references in order to make such decisions. Consequently he will add enormously to his knowledge of the general subject involved, and it enables him in many cases to arrive at new results and publish them.* To illustrate this, the writer has collected (probably incomplete) bibliographies on a number of topics in the theory of numbers and distributed copies to various mathematicians whom he assumed would be interested. The writer is well aware of the fact that his own knowledge of each topic increased greatly thereupon, and the publication of a number of his papers was due to this.

In the first part of this paper we discussed the advantages of trying to bring up to date, or nearly up to date, Dickson's *History of the Theory of Numbers*. The first step in such a project should be the collection of *all references on number theory* starting with the year 1920 and then classifying them according to different topics. *In case it turns out that the project of continuing Dickson's work of reviewing each article falls through, then the classified list mentioned, without reports, will still be of immense value to number theorists.*

Let us now consider the physical environment under which a mathematician who undertakes a work of this kind would find most convenient. In this connection perhaps we cannot improve on the conditions surrounding Dickson while he was writing his *History*. He had an office adjoining the Mathematics Library, which fine library was separate from the Main Library at the University of Chicago. Naturally, then, his office was *very quiet*, a fact, of course, which helped him in concentrating on any matter at hand. Also, if he wished to consult or review any mathematical article, all he usually had to do was walk a few steps to locate it. If he did so, and found that he wished to study it at length, he would take the book containing the article to his office and keep it there until he had finished his work on it. If the librarian could not find some book which the records showed should have been in the stacks, he assumed it was on Dickson's desk.

For a mathematician to be in the best position to head a project for preparing a bibliography of the type we have been discussing, he should, of course, already have had considerable experience with the relevant literature. However, not all the men working with him need to be so well prepared. A young mathematician might gradually learn from a more experienced man how to arrange for the proper recognition of the topics treated in a given paper in order to attain adequate classification in the proposed bibliography.

In this MONTHLY, vol. 57, no. 7, Part II, 1950, pp. 1–110, an index of Volumes 1–56 of the MONTHLY, classified according to topics, is given. If I recall correctly, similar indices have been provided by other magazines from time to time. These partial bibliographies will, of course, help in the preparation of a complete one.

The 1958 Parliament of Science, held in March, 1958, Washington, D. C. (*Science*, vol. 127, 1958, p. 856), under the topic "Recommendations Concerning Communication Among Scientists," stated that "It is recommended that research-supporting agencies: . . . (ii) Recognize that monographs, abstracts, indices, reviews, compendia, books, *bibliographies*, and proceedings of symposia are important means of communication and that scientists should be supported in their preparation." (Italics of word in quotation are those of the author.)

E. J. McShane (*Maintaining Communication*, this MONTHLY, vol. 64, 1957, p. 313) treats various aspects of this topic, including the desirability of mathematicians engaged in work of the character done by Dickson in his *History*.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

A DEFINITION OF ORDINAL NUMBERS

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This note presents a new definition of an ordinal number which is simpler in conceptual structure than any other known to the author. The proof that it agrees with Gödel's definition [1] (which involves the concept of well-ordering) can be given in a three-line transfinite induction. However, the following development has the advantage of requiring no preparation beyond the axioms of set theory and some acquaintance with the axiomatic method, and it leads naturally to the idea of well-ordering.

A set X is said to be *transitive* if every element of X is a subset of X. We have at once

THEOREM 1. *A set X is transitive if any of the following are true. X is a union of transitive sets; or X is an intersection of transitive sets; or $X = Y \cup \{Y\}$, where Y is transitive.*

Proof. If X is a union of transitive sets S_α, each element x of X is an element of some S_α, and $x \subset S_\alpha \subset X$. In the case of an intersection x is an element of every S_α, and $x \subset \cap S_\alpha = X$. The last case is also routine.

A set X is called an *ordinal number* if every transitive proper subset of X is an element of X. Let us call a set S *untangled* if S has no element which is an element of itself.

THEOREM 2. *Every ordinal number is transitive and untangled.*

Proof. Let X be an ordinal number. Since a union of untangled sets is clearly untangled, if we let X' be the union of all transitive untangled subsets of X we see that X' itself will be transitive and untangled; in fact, X' is the largest untangled transitive subset of X. Clearly what we wish to prove is that $X = X'$. But if we suppose $X' \neq X$, then $X' \in X$ (by definition of *ordinal*), and so $X' \cup \{X'\}$ is a subset of X which is transitive (by Theorem 1) and untangled (since $X' \notin X'$). By the maximality of X' it follows that $X' \cup \{X'\} \subset X'$, which implies the contradiction $X' \in X'$.

THEOREM 3. *For any two ordinal numbers, X, Y, one is a subset of the other. In fact, if $X \neq Y$, one is an element of the other.*

Proof. $X \cap Y$ is a transitive subset of X and of Y. It is not an element of both, for this would make $X \cap Y \in X \cap Y$. Thus $X \cap Y$ is X or Y, and if it is equal to just one of them, it is an element of the other.

THEOREM 4. *Every transitive set of ordinal numbers is an ordinal number.*

Proof. Let S be a transitive set of ordinals and Y a transitive proper subset of S. Consider some element s of $S-Y$. For any $y\in Y$, $s\neq y$ and $s\notin y$ (since Y is transitive). Therefore $y\in s$ for each y, and the transitive set Y is a subset of the ordinal number s. Either $Y=s$ or $Y\in s$; in either case, $Y\in S$, as required.

THEOREM 5. *Every element of an ordinal number is an ordinal number.*

Proof. For any ordinal number X, let X' be the union of all transitive sets of ordinal numbers contained in X. X' is of course a transitive subset of X, and by Theorem 4, X' is an ordinal. Hence $X'\in X$ would imply that the transitive set $X'\cup\{X'\}$ is a subset of X', contrary to Theorem 2. Therefore X' is not a proper subset of X, $X=X'$.

THEOREM 6. *The intersection of any nonempty set of ordinal numbers is an ordinal number and an element of the set.*

Proof. The intersection of $\{X_\alpha\}$ is a transitive set of ordinals, hence an ordinal, X. If $X\neq X_\alpha$ for all α we should have $X\in X_\alpha$ for all α, whence $X\in X$.

This completes the contact with the usual treatments.

Reference

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DEVELOPMENTS OF $\sin^a x$ INTO FOURIER SERIES

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The function $\sin^a x$ is well defined and continuous for all real values of a and all x on the interval I_0: $0<x<\pi$. If $a>-1$ it satisfies the conditions of Dirichlet on $0\leqq x\leqq\pi$ for a development into a Fourier series. For that purpose we define the following two continuations of $\sin^a x$:

$$F(x+k\pi, a) = (-1)^k \sin^a x, \tag{1}$$

$$G(x+k\pi, a) = \sin^a x, \tag{2}$$

for all $x\in I_0$ and all $k=0, \pm 1, \cdots$.

For $x=0$ and all $k=0, \pm 1, \cdots$ we define $F(k\pi, a)=G(k\pi, a)=0$. Thus F and G are continuous for all x and $a>0$ and discontinuous at $x=k\pi$ if $a\leqq 0$. Clearly $F(x, a)=(-1)^k|\sin x|^a$ and $G(x, a)=|\sin x|^a$ for all x on the interval I_k: $k\pi<x<(k+1)\pi$.

Since $\sin^a(\pi/2+x)=\sin^a(\pi/2-x)$ for $0\leqq x<\pi/2$ and $F(-x, a)=-F(x, a)$, we must have

$$F(x, a) = \sum_{n=0}^{\infty} A_n(a) \sin(2n+1)x, \tag{3}$$

and, similarly,

$$G(x, a) = \frac{B_0}{2} + \sum_{n=0}^{\infty} B_n(a) \cos 2nx. \tag{4}$$

The integrals

$$A_n(a) = \frac{1}{\pi}\int_0^{2\pi} F(x, a) \sin (2n + 1)x\, dx = \frac{2}{\pi}\int_0^{\pi} \sin^a x \sin (2n + 1)x\, dx,$$

$$B_n(a) = \frac{1}{\pi}\int_0^{2\pi} G(x, a) \cos 2nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin^a x \cos 2nx\, dx,$$

may be evaluated using de Moivre's formulas*:

$$A_n(a) = \frac{2}{\pi}\sum_{m=0}^{n} (-1)^m \binom{2n + 1}{2m + 1} \int_0^{\pi} \sin^{a+2m+1} x \cos^{2(n-m)} x\, dx, \tag{5}$$

$$B_n(a) = \frac{2}{\pi}\sum_{m=0}^{n} (-1)^m \binom{2n}{2m} \int_0^{\pi} \sin^{a+2m} x \cos^{2(n-m)} x\, dx. \tag{6}$$

All the integrals are of the form $\int_0^{\pi} \sin^{\alpha} x \cos^{2\lambda} x\, dx$, $\lambda = 0, 1, \cdots, \alpha > -1$, which, putting $t = \sin^2 x$, becomes

$$\int_0^{\pi} \sin^{\alpha} x \cos^{2\lambda} x\, dx = 2\int_0^{\pi/2} \sin^{\alpha} x \cos^{2\lambda} x\, dx$$

$$= \int_0^1 t^{(\alpha+1)/2-1}(1 - t)^{(2\lambda+1)/2-1} dt = \frac{\Gamma(\frac{1}{2}\alpha + \frac{1}{2})\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1 + \frac{1}{2}\alpha)}.$$

Substituting these values in (5) and (6) and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get:

$$A_n(a) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1 + \frac{1}{2}a)}{\Gamma(n + 1 + \frac{1}{2}a + \frac{1}{2})} P_n(a), \qquad B_n(a) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2})}{\Gamma(n + 1 + \frac{1}{2}a)} Q_n(a),$$

where

$$P_n(a) = \sum_{m=0}^{n} (-1)^m \binom{2n + 1}{2m + 1} \frac{\Gamma(n - m + \frac{1}{2})\Gamma(m + 1 + \frac{1}{2}a)}{\Gamma(\frac{1}{2})\Gamma(1 + \frac{1}{2}a)}, \tag{7}$$

$$Q_n(a) = \sum_{m=0}^{n} (-1)^m \binom{2n}{2m} \frac{\Gamma(n - m + \frac{1}{2})\Gamma(m + \frac{1}{2}a + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2})}, \tag{8}$$

are polynomials of degree n in a.

To represent these polynomials, we observe that (3) and (4) contain the inverse formulas of de Moivre:

$$\sin^{2m+1} x = \sum_{n=0}^{m} \frac{(-1)^n}{2^{2m}} \binom{2m + 1}{m + n} \sin (2n + 1)x \equiv \sum_{n=0}^{\infty} A_n(2m + 1) \sin (2n + 1)x,$$

* *I.e.*, formulas which represent $\sin kx$ and $\cos kx$ as polynomials in $\sin x$ and $\cos x$.

$$\sin^{2m} x = \sum_{n=0}^{m} \frac{(-1)^n}{2^{2m-1}} \binom{2m}{m+n} \cos 2nx \equiv \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n(2m) \cos 2nx.$$

From these formulas we have, for $n>0$, $A_n(2m+1)=0$ and $B_n(2m)=0$, $m=0, 1, \cdots, n-1$. Hence

$$P_n(a) = p(a-1)(a-3)\cdots(a-(2n-1)),$$
$$Q_n(a) = qa(a-2)\cdots(a-(2n-2)).$$

From (7) and (8) we determine p and q as

$$p = \lim_{n\to\infty} \frac{P_n(a)}{a^n} = \frac{(-1)^n}{2^n} = \lim_{n\to\infty} \frac{Q_n(a)}{a^n} = q.$$

Thus we obtain

$$P_n(a) = (n - (\tfrac{1}{2}a + \tfrac{1}{2}))(n - 1 - (\tfrac{1}{2}a + \tfrac{1}{2})) \cdots (1 - (\tfrac{1}{2}a + \tfrac{1}{2}))$$
$$= \frac{\Gamma(n + 1 - (\tfrac{1}{2}a + \tfrac{1}{2}))}{\Gamma(1 - (\tfrac{1}{2}a + \tfrac{1}{2}))},$$

$$Q_n(a) = (n - 1 - \tfrac{1}{2}a)(n - 2 - \tfrac{1}{2}a) \cdots (-\tfrac{1}{2}a) = \frac{\Gamma(n - \tfrac{1}{2}a)}{\Gamma(-\tfrac{1}{2}a)},$$

with $P_0(a)=Q_0(a)=1$; *i.e.*, the representation of $P_n(a)$ and $Q_n(a)$ by a quotient of Γ-functions also holds for $n=0$. Then, for $0<x<\pi$,

$$\sin^a x = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1 + \tfrac{1}{2}a)}{\Gamma(1 - (\tfrac{1}{2}a + \tfrac{1}{2}))} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - (\tfrac{1}{2}a + \tfrac{1}{2}))}{\Gamma(n + 1 + (\tfrac{1}{2}a + \tfrac{1}{2}))} \sin (2n+1)x, \tag{9}$$

or

$$\sin^a x = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\tfrac{1}{2}a + \tfrac{1}{2})}{\Gamma(-\tfrac{1}{2}a)} \left[\frac{\Gamma(-\tfrac{1}{2}a)}{2\Gamma(1 + \tfrac{1}{2}a)} + \sum_{n=1}^{\infty} \frac{\Gamma(n - \tfrac{1}{2}a)}{\Gamma(n + 1 + \tfrac{1}{2}a)} \right] \cos 2nx. \tag{10}$$

For all x on the interval $k\pi<x<(k+1)\pi$, where $k=0, \pm 1, \cdots$, the right sides of (9) and (10) represent $(-1)^k|\sin x|^a$ and $|\sin x|^a$ resectively.

Remark. Denoting by α and β arbitrary real numbers and considering the linear combination

$$Y(x, a) = \alpha G(x, a) + \beta F(x, a) = \begin{cases} (\alpha + \beta) \sin^a x, & 0 < x < \pi, \\ (\alpha - \beta) |\sin x|^a, & -\pi < x < 0 \end{cases}$$

$Y(x, a)$ represents all possible continuations of solutions of the differential equation $y'=ay \cot x$ in the interval $-\pi<x<\pi$. Especially for $\alpha+\beta=0$, an interesting representation of zero in the interval $0<x<\pi$ is obtained.

NOTE ON HAMILTON CIRCUITS*

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In a recent note, D. J. Newman† proved the following result:

Any graph with $2n$ vertices each of order not less than n must contain a $2n$-gon.

The object of the present observations is to point out that the argument used by Newman can be used to obtain somewhat more general results on *Hamilton circuits*, that is, closed paths in graphs passing through every vertex once.

Let G be a finite graph without loops or multiple edges. The m vertices of G we arrange in some cyclic order

$$a_0, a_1, \cdots, a_{m-1}, a_0. \tag{1}$$

There is a *gap* between two vertices a_k and a_{k+1} in (1) if there is no edge connecting them. When a Hamilton circuit exists it is possible to find a cyclic order (1) without gaps.

Assume now that (1) is a cyclic arrangement of the vertices with a minimal number of gaps such that there is a gap between a_0 and a_1. Let (a_0, a_i) be one of the $\rho(a_0)$ edges from a_0. Then there can be no edge (a_1, a_{i+1}) because it would lead to a new order

$$a_0, a_i, a_{i-1}, \cdots, a_1, a_{i+1}, \cdots, a_{m-1}, a_0$$

of the vertices with one gap less.

We conclude that if (1) is an arrangement with a minimal number of gaps, none of the $\rho(a_0)$ vertices a_i connected by an edge to a_0 can be followed by any of the $\rho(a_1)$ vertices connected to a_1. Thus there are at least as many vertices not connected to a_1 as there are connected to a_0. It follows that $m-1-\rho(a_1) \geqq \rho(a_0)$. This gives:

THEOREM 1. *Let* (1) *be a cyclic order of the vertices in G with a minimal number of gaps. If there is a gap between a pair of vertices a_k and a_{k+1} then the corresponding local degrees satisfy $\rho(a_k)+\rho(a_{k+1}) \leqq m-1$.*

An immediate consequence is:

THEOREM 2. *Let G be a graph with m vertices such that for any pair of vertices a and b not connected by an edge the local degrees satisfy the condition $\rho(a)+\rho(b) \geqq m$. Then G has a Hamilton circuit.*

In particular G has a Hamilton circuit if $\rho(a) \geqq m/2$ for every vertex.

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* Research supported by N.S.F. grant.

† A problem in graph theory, this MONTHLY, vol. 65, 1958, p. 611.

A BOOLEAN GEOMETRY FOR THE INTEGERS

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Metric spaces in which the distances are not real numbers have been studied by several people ([1], [2], [3], [4], [5]). The title of this note refers to the metric properties of the ring J of rational integers with respect to a distance function with values in a certain Boolean algebra $\mathfrak{B}$. This Boolean algebra is isomorphic to the algebra of subsets of a countable set. It is more convenient here, however, to define $\mathfrak{B}$ in the following way: denote by $\mathfrak{B}_2$ the Boolean algebra with two elements 0 and 1, then $\mathfrak{B}$ is the Boolean algebra whose elements are sequences $\{a_i\}$ of elements of $\mathfrak{B}_2$, with union and intersection defined componentwise.

Let p_i be the ith positive prime integer, and define a mapping $x \to \phi(x)$ of J into $\mathfrak{B}$ as follows: for $n \in J$, $\phi(n) = \{a_i\}$, where a_i is 0 or 1 according as p_i is or is not a factor of n. It is easily seen that the set of images, $\mathfrak{B}^* = \phi(J)$ is a sublattice of $\mathfrak{B}$, and that the mapping $x \to \phi(x)$ has the following properties:

$$\phi(n) = \phi(0) \text{ if and only if } n = 0, \tag{1}$$

$$\phi(n + m) \subseteq \phi(n) \cup \phi(m), \tag{2}$$

$$\phi(n \cdot m) = \phi(n) \cap \phi(m), \tag{3}$$

$$\phi(n) = \phi(1) \text{ if and only if } n = \pm 1. \tag{4}$$

Conversely, it can be shown that if ϕ is a mapping of J into a lattice $\mathfrak{L}$ satisfying these four conditions, then the set $\mathfrak{L}^* = \phi(J)$ is a sublattice of $\mathfrak{L}$ isomorphic to the lattice $\mathfrak{B}^*$ defined above.

Any ring R together with a mapping $x \to \phi(x)$ into a Boolean algebra $\mathfrak{A}$ satisfying (1), (2), and (3) is called a *Boolean-valued ring*. The mapping ϕ is called a valuation and $\mathfrak{A}$ is called a *valuation algebra*. If R is a Boolean-valued ring then the mapping ρ of $R \times R$ into $\mathfrak{A}$ defined by $\rho(x, y) = \phi(y - x)$, is called a distance function on R. It is easily seen that ρ satisfies:

$$\rho(x, y) = \phi(0) \text{ if and only if } x = y, \tag{5}$$

$$\rho(x, y) = \rho(y, x), \tag{6}$$

$$\rho(x, z) \cup \rho(z, y) \supseteq \rho(x, y). \tag{7}$$

The ring R together with this distance function is called a *Boolean space*. Thus, the ring of integers is a Boolean space with respect to the valuation ϕ defined above.

The first of the following two theorems characterizes the ring of integers as the only Boolean-valued ring with valuation algebra $\mathfrak{B}$ which is congruent to this Boolean space of the integers. The second theorem shows that the only isometries of this Boolean space of the integers are reflections and translations.

THEOREM 1. *Let R be a Boolean-valued ring with valuation algebra $\mathfrak{B}$ and valuation ψ and let ϕ be the valuation defined above for J. If $x \to g(x)$ is a* 1-1 *mapping*

of R onto J with $\psi(y-x)=\phi(g(y)-g(x))$ for all x, y in R, then R is isomorphic to J.

Proof. Let $f(x)=g(x)-g(0)$; then $f(x)$ is a 1-1 mapping of R onto J, with $f(0)=0$. Further, $\psi(y-x)=\phi(f(y)-f(x))$ for x, y in R and, in particular,

$$\psi(y) = \phi(f(y)), \qquad y \in R. \tag{8}$$

Next, note that R contains no proper divisors of zero. For if $xy=0$, then $\phi(0)=\psi(xy)=\psi(x)\cap\psi(y)=\phi(f(x))\cap\phi(f(y))=\phi(f(x)\cdot f(y))$. This implies $f(x)\cdot f(y)=0$, $f(x)=0$ or $f(y)=0$, and hence either x or y is zero. Now, choose u, v in R so that $f(u)=1$ and $f(v)=-1$. It follows from (4) and (8) that $f(u^2)=\pm 1$, $f(v^2)=\pm 1$, and $f(uv)=\pm 1$. From the first two of these last three equations, $f(u^2)=\pm f(v^2)$. If $f(u^2)=-f(v^2)$ then either $f(u^2)=f(uv)$ or $f(v^2)=f(uv)$, both of which lead to contradictions. Consequently $f(u^2)=f(v^2)=\pm 1$, and either $u^2=u$ or $v^2=v$. Thus, either u or v is an idempotent and hence an identity in R. Denote by e this identity element. Since $f(e)=\pm 1$ there is no loss in generality in assuming that $f(e)=1$. For, if $f(e)=-1$ the mapping $x\to f'(x)=-f(x)$ satisfies the hypothesis of the theorem and $f'(e)=1$. Suppose that $e+e=0$, and let y be the element of R such that $f(y)=-1$. Then $y\neq e$, but $\phi(f(y^2))=\psi(y^2)=\psi(y)=\phi(f(y))=\phi(-1)=\phi(1)$, so that $f(y^2)=\pm 1$. Thus, either $y^2=y$, a contradiction or $y^2=e=e^2$, which implies $(y-e)(y+e)=0$, also a contradiction. Hence $e+e\neq 0$, $e\neq -e$, and since $\phi(f(-e))=\psi(-e)\cap\psi(-e)=\psi(-e)=\psi(e)=\phi(1)$, it follows that $f(-e)=-1$. By two finite induction arguments it may be shown that $f(ne)=n$ and $f[(-n)e]=-n$, for every positive integer n. It then follows readily that $x\to f(x)$ is an isomorphism of R onto J.

A 1-1 mapping $x\to m(x)$ of J onto J is called an *isometry* if for all x, y in J, $\phi(y-x)=\phi(m(y)-m(x))$.

THEOREM 2. *If $x\to m(x)$ is an isometry of J, then for all x either $m(x)=x+m(0)$ or $m(x)=-x+m(0)$.*

Proof. Let $f(x)=m(x)-m(0)$, so that $f(0)=0$. Also the mapping $x\to f(x)$ is an isometry. Thus $\phi(f(1))=\phi(1)$, which implies $f(1)=\pm 1$. Suppose first that $f(1)=1$. As in the proof of Theorem 1, two finite induction arguments show that $f(n)=n$ and $f(-n)=-n$ for every positive integer n. Thus $x=f(x)=m(x)-m(0)$, for every x in J and hence $m(x)=x+m(0)$. If $f(1)=-1$, then by the preceding argument $f'(x)=x$ for every x in J, where $f'(x)=-f(x)$. Thus $-x=f(x)=m(x)-m(0)$ or $m(x)=-x+m(0)$. This completes the proof.

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THE DOUBLE DIXIE CUP PROBLEM

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The familiar childhood occupation of obtaining a complete set of pictures of baseball players, movie stars, etc., which appear on the covers of dixie cups raises some interesting questions. One, which has already been answered, is the "single dixie cup problem," that of determining the expected number, $E(n)$, of dixie cups which must be purchased before a complete set of n pictures is obtained: $E(n)=n(1+1/2+\cdots+1/n)$ ([1] p. 213).

Some time ago W. Weissblum asked how long, on the average, it would take to obtain *two* complete sets of n pictures. This corresponds to the situation observed when two tots collect cooperatively, *i.e.*, "trading" takes place.

This "double dixie cup" problem cannot be handled by the same device used for the problem of the single set and in this paper we find a new method which allows us to write down the solution, $E_m(n)$, (as an easily evaluated definite integral) for the problem of collecting m sets.

For m fixed and n large the expected number of dixie cups turns out to be $n(\log n+(m-1)\log\log n+o(1))$. Thus, although the first set "costs" $n\log n$, all further sets cost $n\log\log n$.

Suppose m sets are desired. Let p_i be the probability of failure of obtaining m sets up to and including the purchase of the ith dixie cup. Then the expected number of dixie cups $E_m(n)=\sum_{i=0}^{\infty} p_i$, by a well-known argument ([1] p. 211). Now $p_i=N_i/n^i$ where N_i is the number of ways that the purchase of i dixie cups can fail to yield m copies of each of the n pictures in the set. If we represent the pictures by $x_1,\cdots,x_n$, then N_i is simply $(x_1+\cdots+x_n)^i$ expanded and evaluated at $(1,\cdots,1)$ after all the terms have been removed which have each exponent for each variable larger than $m-1$.

Now consider m fixed and introduce the following notation. If $P(x_1,\cdots,x_n)$ is a polynomial or power series we define $\{P(x_1,\cdots,x_n)\}$ to be the polynomial, or series, resulting when all terms having all exponents $\geqq m$ have been removed. In terms of this notation p_i is $\{(x_1+\cdots+x_n)^i\}/n^i$ evaluated at $x_1=\cdots=x_n=1$.

If we now make the definition

$$S_m(t)=\sum_{k<m}\frac{t^k}{k!} \tag{1}$$

and consider the expression

$$F=\exp(x_1+\cdots+x_n)-(e^{x_1}-S_m(x_1))\cdots(e^{x_n}-S_m(x_n)), \tag{2}$$

it is easily seen that F has no terms with all exponents $\geqq m$; but F does not have all terms of $\exp(x_1+\cdots+x_n)$ with at least one exponent $<m$. We conclude that

$$F = \{\exp(x_1 + \cdots + x_n)\} = \sum \{(x_1 + \cdots + x_n)^i\}/i!. \tag{3}$$

By contrast, we have seen that

$$E_m(n) = \sum p_i = \sum \{(x_1 + \cdots + x_n)^i\}/n^i \tag{4}$$

at $x_1 = \cdots = x_n = 1$, and so all we need now is a method for replacing $1/i!$ by $1/n^i$. This is afforded by the identity

$$n\int_0^\infty \frac{t^i}{i!}e^{-nt}dt = \frac{1}{n^i}$$

and the result is

$$n\int_0^\infty [\exp(x_1 + \cdots + x_n)^t - (e^{x_1 t} - S_m(tx_1)) \cdots (e^{x_n t} - S_m(tx_n)]e^{-nt}dt. \tag{5}$$

Setting $x_1 = \cdots = x_n = 1$, finally, gives, by (4) and (5),

THEOREM 1. $n\int_0^\infty [1-(1-S_m(t)e^{-t})^n]dt = E_m(n)$.

This is the solution and it is readily integrable for small m and n. It is perhaps worthwhile to mention that if only a particular k of the pictures are desired the expected number is:

$$n\int_0^\infty 1 - \left(1 - e^{-t}\left(1 + t \cdots + \frac{t^{m-1}}{(m-1)!}\right)\right)^k dt.$$

This may also be easily generalized to the case where m_k copies of the kth pictures are desired.

For large m, by the law of large numbers, the number required is asymptotic to mn. It remains to obtain the asymptotic form for large n.

We now prove

THEOREM 2. $E_m(n) = n[\log n + (m-1)\ \text{loglog}\ n + C_m + o(1)]$ *for* m *fixed and* $n \to \infty$.

It suffices to prove that

$$\frac{E_m(n+1)}{n+1} - \frac{E_m(n)}{n} = \frac{1}{n+1} + \frac{m-1}{n\log n} + \lambda_n,$$

where $\sum|\lambda_n| < \infty$.

Now, by Theorem 1,

$$\frac{E_m(n+1)}{n+1} - \frac{E_m(n)}{n} = \int_0^\infty e^{-t}S_m(t)[1 - e^{-t}S_m(t)]^n dt$$

and, changing variables by

$$1 - e^{-t}S_m(t) = x, \tag{6}$$

we have $dx = e^{-t}[(t^{m-1})/(m-1)!]dt$ and so the above is equal to

$$(7)\qquad \int_0^1 x^n S_m(t)\frac{(m-1)!}{t^{m-1}}\,dx = \int_0^1 x^n\left[1+\frac{m-1}{t}+\frac{(m-1)(m-2)}{t^2}+\cdots\right]dx.$$

[$t=t(x)$ is of course defined by (6)].

We now show that

$$(A)\qquad \text{For } 1<k<m,\quad \sum_{n=1}^{\infty}\int_0^1 \frac{x^n}{t^k}\,dx<\infty;$$

$$(B)\qquad \int_0^1 \frac{x^n}{t}\,dx = \frac{1}{n\log n}+\alpha n,\quad \text{where}\quad \sum|\alpha n|<\infty;$$

and, by (7), these will suffice to prove our theorem.

Proof of (A). By (6) we have $x=1-e^{-t}S_m(t)\leqq 1-e^{-t}$ so that,

$$(8)\qquad t\geqq \log\frac{1}{1-x}.$$

On the other hand

$$x=1-e^{-t}S_m(t)=\int_0^t e^{-u}\frac{u^{m-1}}{(m-1)!}\,du\leqq\int_0^t u^{m-1}du\leqq t^m,$$

so that

$$(9)\qquad t\geqq x^{1/m}.$$

Now, the infinite series given in (A) is equal to

$$\int_0^1\frac{1}{1-x}\frac{dx}{t^k}=\int_0^{1/2}\frac{1}{1-x}\frac{dx}{t^k}+\int_{1/2}^1\frac{1}{1-x}\frac{dx}{t^k}.$$

The first of the integrals on the right is finite by virtue of (9) and the fact that $k<m$, while the second integral is finite by (8) and the fact that $k>1$. This proves (A).

Proof of (B). By (8) we obtain

$$t\geqq x+\frac{x^2}{2}+\cdots\frac{x^r}{r}\geqq x^r\left[1+\frac{1}{2}+\cdots\frac{1}{r}\right]\geqq x^r\log r,$$

and so

$$(10)\qquad \int_0^1\frac{x^n}{t}\,dx\leqq\int_0^1\frac{x^{n-r}}{\log r}\,dx=\frac{1}{(n-r+1)\log r}.$$

Now let $u\geqq 1$ be a parameter and set

$$(11)\qquad a=1-S_m(u)e^{-u}.$$

We have

$$\int_0^1 \frac{x^n}{t}\,dx \geqq \int_0^u \frac{x^n}{t}\,dx \geqq \frac{1}{u}\int_0^u x^n dx$$

$$\geqq \frac{1}{u}\left[\int_0^1 x^n dx - \int_u^1 dx\right] = \frac{1}{(n+1)u} - \frac{S_m(u)e^{-u}}{u} \qquad \text{by (11).}$$

Now note that by (1) and the fact that $u \geqq 1$, we obtain $S_m(u) \leqq eu^{m-1}$. Combining this with the previous inequality gives

$$\int_0^1 \frac{x^n}{t}\,dx \geqq \frac{1}{(n+1)u} - u^{m-2}e^{1-u}. \tag{12}$$

If we now set $r = [n/\log n]$ in (10) and $u = \log n + m \log\log n$ in (12) we obtain

$$\frac{1}{n\log n} - \frac{C\log\log n}{n\log^2 n} \leqq \int_0^1 \frac{x^n}{t}\,dx \leqq \frac{1}{n\log n} + \frac{C\log\log n}{n\log^2 n},$$

and this completes the proof since $\sum(\log\log n)/(n\log^2 n) < \infty$.

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THE ANTICENTER OF A GROUP

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In contrast to the center of a group G we will define the rim of G, denoted by $R(G)$, to be the set of elements of G which permute with no elements of G except in trivial cases. We will define the anticenter of G, denoted by $AC(G)$, as the set of products of elements in $R(G)$. More precisely:

DEFINITION 1. $R(G) \equiv \{a \mid ab = ba$ *implies* $\exists\, c \in G$ *such that* $a = c^i$, $b = c^j$ *for some integers* i *and* $j.\}$

DEFINITION 2. $AC(G) \equiv \{a_1 \cdots a_n \mid a_i \in R(G).\}$

LEMMA 1. *The identity* e *of* G *belongs to* $R(G)$.

Proof. $eb = be$ implies $e = b^0$, $b = b^1$.

LEMMA 2. *If* $a \in R(G)$ *then* $a^{-1} \in R(G)$.

Proof. Suppose $a^{-1}b = ba^{-1}$. Then $ab = ba$ and hence $a = c^i$ and $b = c^j$ or $b = c^j$ and $a^{-1} = c^{-i}$.

LEMMA 3. *If* $a \in R(G)$ *then* $b^{-1}ab \in R(G)$ *for all* $b \in G$.

Proof. Let $b^{-1}abx = xb^{-1}ab$ for some x in G. Then $abxb^{-1} = bxb^{-1}a$. Hence $a = c^i$ and $bxb^{-1} = c^j$ or $b^{-1}ab = b^{-1}c^ib = (b^{-1}cb)^i$ and $x = b^{-1}c^ib = (b^{-1}cb)^j$. This proves $b^{-1}ab \in R(G)$.

THEOREM 1. *$AC(G)$ is a normal subgroup of G.*

Proof. $AC(G)$ is nonempty, closed under multiplication, and if u is in $AC(G)$ then u^{-1} is in $AC(G)$ as a consequence of Lemma 2. Hence $AC(G)$ is a subgroup of G. To show that $AC(G)$ is normal, let $a_1 \cdot \cdot \cdot a_n \in AC(G)$. Then $b^{-1}a_1 \cdot \cdot \cdot a_n b = (b^{-1}a_1 b) \cdot \cdot \cdot (b^{-1}a_n b)$ and this belongs to $AC(G)$ since by Lemma 3, $b^{-1}a_i b$ belongs to $R(G)$.

THEOREM 2. *If G is cyclic then $R(G) = AC(G) = G$.*

Proof. Since $R(G) \subset AC(G) \subset G$, it suffices to show that $G \subset R(G)$. Let b generate G and let $a \in G$. Then $ac = ca$ implies a and c are powers of b and hence $a \in R(G)$.

THEOREM 3. *If G is finite and abelian, then $R(G) = AC(G) = G$ if and only if G is cyclic.*

Proof. The sufficiency is a consequence of Theorem 2. To show the necessity let $G: a_1, \cdot \cdot \cdot, a_n$. Now $a_1 a_2 = a_2 a_1$ and hence a_1 and a_2 are both powers of some c_1. But $a_3 c_1 = c_1 a_3$ and hence a_3 and c_1 are powers of some c_2. Hence a_1, a_2, and a_3 are powers of c_2. By induction $a_1, \cdot \cdot \cdot, a_n$ are powers of some c_{n-1} and hence G is cyclic.

THEOREM 4. *If G is abelian, then $R(G) = AC(G)$.*

Proof. It suffices to show that $R(G)$ is closed under multiplication. Let a and b belong to $R(G)$. Suppose that $(ab)x = x(ab)$. Then $a(bx) = (bx)a$ and hence $a = c^i$ and $bx = c^j$. But $bc = cb$ and this implies $b = d^k$ and $c = d^s$. Hence $a = c^i = d^{is}$ and $b = d^k$. Thus $ab = d^{k+is}$ and $x = b^{-1}c^j = d^{-k}d^{sj} = d^{sj-k}$. Thus $ab \in R(G)$.

THEOREM 5. *If G is isomorphic to G^*, then $AC(G)$ is isomorphic to $AC(G^*)$.*

The reader will easily construct a proof.

Example 1. Let $G \equiv \{e, a, b, c\}$ in which $a^2 = b^2 = c^2 = e$ and $ac = ca = b$, $ab = ba = c$, and $cb = bc = a$. Then $e = R(G) = AC(G)$.

Example 2. Let G be represented by I, A, A^2, B, BA, BA^2, where $A^3 = I = B^2$ and $AB = BA^2$. The list of pairs which permute are: $IX = XI$, $XX = XX$ and $A^2A = AA^2$, where X is arbitrary in the group. In each case the permuting is trivial and hence $R(G) = AC(G) = G$.

Example 3. Let $G \equiv \{I, R, R^2, R^3, D_1, D_2, H, V\}$ be the group of symmetries of the square where $R = (1234)$, $D_1 = (13)$, $D_2 = (24)$, $H = (23)(14)$, and $V = (12)(34)$. The list of pairs that permute is as follows: $XI = IX$, $XX = XX$ for all X in G, $R^2R = RR^2$, $R^3R = RR^3$, $R^3R^2 = R^2R^3$, $D_1R^2 = R^2D_1$, $D_2R^2 = R^2D_2$, $D_1D_2 = D_2D_1$, $VR^2 = R^2V$, $VH = HV$, and finally, $HR^2 = R^2H$. From this list it is clear that I, R, and R^3 are in $R(G)$. We will now show that no other element in G is in $R(G)$. For $VR^2 = R^2V$, but $V \neq D_1^i, D_2^i, H^i, R^i, I$ for all i and $R^2 \neq V^i$ for all i. Hence $R^2 \notin R(G)$ and $V \notin R(G)$. Also $D_1D_2 = D_2D_1 = R^2$, but $D_2 \neq D_1^i, H^i$,

V^i, R^i, I and $D_1 \neq D_2^i$ for all i. Hence $D_1 \notin R(G)$ and $D_2 \notin R(G)$. Finally $HR^2 = V = R^2H$. But $H \neq V^i$, R^i, D_1^i, D_2^i, I and $R^2 \neq H^i$ for all i. Hence $H \notin R(G)$. Thus $AC(G)$: I, R, R^2, and R^3.

The converses of Theorem 2 and Theorem 4 are false as Example 2 indicates. Finally the $AC(G)$ is, in general nontrivial, as is demonstrated by Example 3.

In a later paper, the author hopes to obtain results concerning $G/AC(G)$, some relations between the center and the anticenter, and to explore more fully the effect of isomorphism and homomorphism on the anticenter.

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CLASSROOM NOTES

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A DERIVATION OF n-DIMENSIONAL SPHERICAL COORDINATES

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An instructive example in linear algebra is the derivation of n-dimensional spherical coordinates without appealing to geometric intuition. The method of derivation is based on concepts from linear algebra; namely, bases of a vector space, scalar product, angle between vectors and projection of a vector onto a subspace. Spherical coordinates in n-dimensions are a generalization of the usual three-dimensional spherical coordinates and are particularly useful in evaluating certain integrals taken over the surface of an n-dimensional sphere. Later we shall give an example of such an integration.

Let E_n denote real n-dimensional euclidean space. Vectors in E_n will be denoted by bold-faced letters. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in E_n with components ξ_j and η_j, $j = 1, \cdots, n$, respectively, we define the scalar product of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} \xi_j \eta_j.$$

The nonnegative number $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ is called the norm of $\mathbf{x}$. The angle between $\mathbf{x}$ and $\mathbf{y}$ is defined by $\cos \phi = \mathbf{x} \cdot \mathbf{y} / \|\mathbf{x}\| \|\mathbf{y}\|$, where ϕ is restricted to the range $0 \leqq \phi \leqq \pi$. A set of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ is an orthonormal set in E_n if $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ or 1 accordingly as $i \neq j$ or $i = j$. Any set of n orthonormal vectors forms a basis for E_n.

Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be any orthonormal basis in E_n. Let $\mathbf{x}$ be any vector on the n-dimensional sphere of radius r about the origin, that is, $\|\mathbf{x}\| = r$. If $\mathbf{x} = \sum_{i=1}^{n} \xi_i \mathbf{e}_i$ then $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} \xi_i^2$. If θ_i is the angle between $\mathbf{x}$ and $\mathbf{e}_i$ then $\xi_i = \mathbf{x} \cdot \mathbf{e}_i = r \cos \theta_i$. Hence $\mathbf{x} = \sum_{i=1}^{n} r \cos \theta_i \mathbf{e}_i$ and $\mathbf{x}$ can be specified by giving its length r and the n angles θ_i. But since $r^2 = \mathbf{x} \cdot \mathbf{x} = r^2 \sum_{i=1}^{n} \cos^2 \theta_i$ we see that the θ_i are not independent of each other. Spherical coordinates in n-dimensions show us how to pick out $n-1$ angles $\phi_1, \cdots, \phi_{n-2}, \theta$ which are independent of each other and which, when combined with the norm r, completely describe the vector $\mathbf{x}$ with respect to the given orthonormal basis.

Derivation of the coordinates. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ and $\mathbf{x}$ be as above. Let ϕ_1 be the angle between $\mathbf{x}$ and $\mathbf{e}_1$, $0 \leqq \phi_1 \leqq \pi$. Then $\xi_1 = \mathbf{x} \cdot \mathbf{e}_1 = r \cos \phi_1$ and

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + \sum_{j=2}^{n} \xi_j \mathbf{e}_j.$$

Now

$$r^2 = \|\mathbf{x}\|^2 = r^2 \cos^2 \phi_1 + \sum_{j=2}^{n} \xi_j^2 \quad \text{or} \quad \sum_{j=2}^{n} \xi_j^2 = r^2 \sin^2 \phi_1.$$

Setting $\xi_j = \alpha_j r \sin \phi_1$, $j = 2, \cdots, n$, we have

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \sum_{j=2}^{n} \alpha_j \mathbf{e}_j,$$

where $\sum_{j=2}^{n} \alpha_j^2 = 1$. (If ϕ_1 is 0 or π, then $\mathbf{x} = \pm r \mathbf{e}_1$.)

Let $\mathbf{u}_2 = \sum_{j=2}^{n} \alpha_j \mathbf{e}_j$. The vector $\mathbf{u}_2$ is a unit vector (that is $\|\mathbf{u}_2\| = 1$) in the direction of the projection of $\mathbf{x}$ onto the $(n-1)$-dimensional subspace spanned by $\mathbf{e}_2, \cdots, \mathbf{e}_n$. If ϕ_2 is the angle between $\mathbf{u}_2$ and $\mathbf{e}_2$ then $\cos \phi_2 = \mathbf{u}_2 \cdot \mathbf{e}_2 = \alpha_2$, $0 \leqq \phi_2 \leqq \pi$, and

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sum_{j=3}^{n} \alpha_j \mathbf{e}_j.$$

Hence,

$$1 = \|\mathbf{u}_2\|^2 = \cos^2 \phi_2 + \sum_{j=3}^{n} \alpha_j^2 \quad \text{or} \quad \sum_{j=3}^{n} \alpha_j^2 = \sin^2 \phi_2.$$

If we set $\alpha_j = \beta_j \sin \phi_2$, $j = 3, \cdots, n$, then

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sin \phi_2 \sum_{j=3}^{n} \beta_j \mathbf{e}_j,$$

where $\sum_{j=3}^{n} \beta_j^2 = 1$. Thus

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \cos \phi_2 \mathbf{e}_2 + r \sin \phi_1 \sin \phi_2 \sum_{j=3}^{n} \beta_j \mathbf{e}_j.$$

In general, let $\mathbf{u}_j$ be the unit vector in the direction of the projection of $\mathbf{x}$ onto the space spanned by $\mathbf{e}_j, \mathbf{e}_{j+1}, \cdots, \mathbf{e}_n$, $j=2, \cdots, n-1$ and let ϕ_{j-1} be the angle between $\mathbf{u}_j$ and $\mathbf{e}_j$, $0 \leqq \phi_j \leqq \pi$, $j=2, \cdots, n-1$. Then

$$\mathbf{x} = \sum_{j=1}^{n-2} r\left(\prod_{k=1}^{j-1} \sin \phi_k\right) \cos \phi_j \mathbf{e}_j + r\left(\prod_{k=1}^{n-2} \sin \phi_k\right) \mathbf{u}_{n-1}.$$

Now $\mathbf{u}_{n-1} = \delta_{n-1}\mathbf{e}_{n-1} + \delta_n \mathbf{e}_n$, where $1 = \|\mathbf{u}_{n-1}\|^2 = \delta_{n-1}^2 + \delta_n^2$. If now we define an angle θ by $\cos \theta = \delta_n$, $\sin \theta = \delta_{n-1}$, we see that $0 \leqq \theta \leqq \pi$ will not suffice since δ_{n-1} can be negative and $\sin \alpha \geqq 0$ for $0 \leqq \theta \leqq \pi$. In order to include all possible combinations of (δ_{n-1}, δ_n) we must have $0 \leqq \theta < 2\pi$.

Thus if $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a given orthonormal basis in E_n and $\mathbf{x}$ is a vector of norm r with components ξ_j with respect to this basis, then

$$(*) \qquad \begin{aligned} \xi_1 &= r \cos \phi_1, \\ \xi_j &= r \cos \phi_j \prod_{k=1}^{j-1} \sin \phi_k \qquad (j = 2, \cdots, n-2), \\ \xi_{n-1} &= r \sin \theta \prod_{k=1}^{n-2} \sin \phi_k, \\ \xi_n &= r \cos \theta \prod_{k=1}^{n-2} \sin \phi_k, \end{aligned}$$

where $0 \leqq \phi_j \leqq \pi$, $j=1, \cdots, n-2$; $0 \leqq \theta < 2\pi$; $0 \leqq r < \infty$.

Application to integration. Let $f(\xi_1, \cdots, \xi_n)$ be a continuous real-valued function defined in E_n which may be written in the form

$$f(\xi_1, \cdots, \xi_n) = g(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, \xi_1^2 + \cdots + \xi_n^2),$$

where the α_i are constants independent of the ξ's. We wish to compute the integral of f over the surface of the n-dimensional sphere of radius r with the origin as center. If $\mathbf{x}$ is the vector with coordinates ξ_j and $\mathbf{a}$ the vector with coordinates α_j (these coordinates being with respect to some given orthonormal basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$) then

$$\int_{\xi_1^2 + \cdots + \xi_n^2 = r^2} f(\xi_1, \cdots, \xi_n) dS = \int_{\|\mathbf{x}\| = r} g(\mathbf{a} \cdot \mathbf{x}, \|\mathbf{x}\|^2) dS,$$

where dS is the surface differential.

Let $\mathbf{a}_1 = \mathbf{a}/\|\mathbf{a}\|$* and choose vectors $\mathbf{a}_2, \cdots, \mathbf{a}_n$ to complete an orthonormal basis in E_n. Let the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{a}_1, \cdots, \mathbf{a}_n$ be $\zeta_1, \cdots, \zeta_n$. Then $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1$. Make the spherical coordinate transformation

* If $\mathbf{a} = \mathbf{0}$, then $\mathbf{a}_1$ may be any unit vector.

given by (*) with ξ_j replaced by ζ_j, $j=1, \cdots, n$.

The Jacobian of the transformation is

$$J = r^{n-1} \prod_{k=1}^{n-2} \sin^k \phi_{n-1-k}.$$

Also, $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1 = r \cos \phi_1$. Thus the integral becomes

$$2\pi r^{n-1} \left[\prod_{k=1}^{n-3} \int_0^\pi \sin^k \phi_{n-1-k} d\phi_{n-1-k} \right] \int_0^\pi g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1$$

$$= \frac{2r^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^\pi g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1.$$

Thus we have reduced the integral over the surface of an n-dimensional sphere to a single integral on the real line. In particular, if $f \equiv 1$, we obtain $[2\pi^{n/2}/\Gamma(n/2)]r^{n-1}$ for the surface area of an n-dimensional sphere of radius r and, integrating from 0 to r, we obtain $[2\pi^{n/2}/(n\Gamma(n/2))]r^n$ for the volume of the sphere.

MATRIX INTEGRATION OF $x^k \exp(-\beta^2 x^2)$

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Let V be the vector space of finite linear combinations of $x^k \exp(-\beta^2 x^2)$, fixed β, $k=0, 1, \cdots$, with basis $\{x^k \exp(-\beta^2 x^2)\}$. Let D be a linear transformation on V which differentiates a vector belonging to V.

Since $(x^k \exp(-\beta^2 x^2))D = kx^{k-1} \exp(-\beta^2 x^2) - 2\beta^2 x^{k+1} \exp(-\beta^2 x^2)$, the matrix of D is

$$\begin{bmatrix} 0 & -2\beta^2 & \cdot & \cdot & \cdots & 0 & \cdots \\ 1 & 0 & -2\beta^2 & \cdot & \cdots & \cdot & \cdots \\ \cdot & 2 & 0 & -2\beta^2 & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & 0 & k & 0 & -2\beta^2 & \cdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

V is closed under D and the kernel of D consists of the zero vector alone. The calculation of D^{-1} may be carried out algebraically, giving an interesting equation for $\int x^k \exp(-\beta^2 x^2) dx$.

Because of the nature of D, D^{-1} may be calculated in four independent steps depending on whether k and j are even or odd, where $\|D^{-1}\| = \|a_{kj}\|$. Using $DD^{-1} = I$, we obtain the following expressions for a_{kj}.

(1) j odd, k odd: $\qquad a_{kj} = 0;$

(2) j even, k odd:
$$a_{kj} = \begin{cases} \dfrac{-(\frac{1}{2}k - \frac{1}{2})!}{2(\frac{1}{2}j)!\beta^{k+1-j}} & j \leqq k-1, \\ 0 & j > k-1; \end{cases}$$

(3) j odd, k even:
$$a_{kj} = \begin{cases} \dfrac{(2\beta^2)^{(j-k-1)/2}}{(k+1)(k+3)\cdots j} & j \geqq k+1, \\ 0 & j < k+1; \end{cases}$$

(4) j even, k even: the elements of D^{-1} are indeterminate with one degree of freedom; choosing a_{00} determines
$$a_{kj} = \frac{\beta^j}{(\frac{1}{2}j)!} \frac{1\cdot 3\cdot 5\cdots(k-1)}{(2\beta^2)^{k/2}} a_{00}.$$

The matrix D thus has an infinite number of inverses (right and left) corresponding to different values of a_{00}. Furthermore, we see that D^{-1} is not properly defined on V since it maps a vector into an infinite linear combination of the basis vectors. We may still use D^{-1} as a computational device to obtain $\int x^k \exp(-\beta^2x^2)dx$.

For k odd, using (1) and (2), we obtain
$$\int x^k \exp(-\beta^2x^2)dx = \frac{-(\frac{1}{2}k - \frac{1}{2})!}{2\beta^{k+1}} \exp(-\beta^2x^2) \sum_{n=0}^{k/2-1/2} \frac{(\beta x)^{2n}}{n!},$$
which is equivalent to that given by Gröbner.* We also easily obtain
$$\int_0^\infty x^k \exp(-\beta^2x^2)dx = \frac{(\frac{1}{2}k - \frac{1}{2})!}{2\beta^{k+1}}, \qquad k \text{ odd}.$$

For k even, using (3) and (4), we obtain the expression†
$$\int x^k \exp(-\beta^2x^2)dx = \exp(-\beta^2x^2)(2\beta^2)^{-(k+1)/2} \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{(k+1)(k+3)\cdots j} + a_{00} \frac{1\cdot 3\cdot 5\cdots(k-1)}{(2\beta^2)^{k/2}} \exp(-\beta^2x^2) \sum_{j=0}^{\infty}{}'' \frac{x^j\beta^j}{(\frac{1}{2}j)!}.$$

Since $\sum'' = \exp(\beta^2x^2)$, the second term in the above expression for the integral reduces to $[a_{00}1\cdot 3\cdot 5\cdots(k-1)]/[(2\beta^2)^{k/2}]$, independent of x.

The choice of a_{00} is only representative of an arbitrary constant of integration, since the term containing it is independent of x. For any D^{-1}, $v \in V$, the series defined by $D^{-1}v$ converge and can be differentiated term by term, and the fundamental theorem of the calculus holds. Hence all of the inverses of D inte-

* Wolfgang Gröbner und Nikolaus Hofreiter, Integraltafel, Erster Teil, Unbestimmte Integrale, Vienna, 1949, p. 109.

† In $\sum'$, the summation is for $j = k+1, k+3, \cdots$; in $\sum''$, for $j = 0, 2, 4, \cdots$.

grate vectors in V.

We then have for k even

$$\int x^k \exp(-\beta^2x^2)dx = \frac{1\cdot3\cdot5\cdots(k-1)}{(2\beta^2)^{(k+1)/2}} \exp(-\beta^2x^2) \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{1\cdot3\cdot5\cdots j}.$$

Since $\int_0^\infty x^k \exp(-\beta^2x^2)dx$ is known, we also obtain an interesting limiting value for a series, namely,

$$\lim_{x\to\infty} \exp(-\beta^2x^2) \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{1\cdot3\cdot5\cdots j} = \frac{\sqrt{\pi}}{2\beta}, \qquad k \text{ even.}$$

NOTE ON THE CLASSICAL CANONICAL FORM OF A MATRIX

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This note concerns some aspects of the proof of the existence and the uniqueness of the classical canonical form of a square matrix without the intervention of invariant factors, elementary divisors, or modules.

As is well known, if the coefficients of A are assumed to lie in an algebraically closed field then by two simple induction proofs* one can show that A is similar to diag $(A_1, A_2, \cdots)$, where each A_i is lower triangular with only one characteristic root, say λ_i. Then one may follow Turnbull and Aitken† to show that A_i is similar to a matrix of the form λ_iI+G, where

$$G = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ b_0 & 0 & & & & & \cdot \\ b_1 & \delta_1 & 0 & & & & \cdot \\ \cdot & & \cdot & 0 & & & \cdot \\ \cdot & & & \cdot & 0 & & \cdot \\ \cdot & & & & \cdot & 0 & \cdot \\ b_{k-2} & \cdot & \cdot & \cdot & \cdot & \delta_{k-2} & 0 \end{bmatrix},$$

δ_j and b_j are 0 or 1, and $\delta_jb_j=0$.

At this point we construct§ the matrix R of a final similarity transformation such that $R^{-1}GR$ is classical, and then prove the uniqueness of the classical form of A by comparing ranks.

1. The matrix R. If q is a maximal integer $\geqq 0$ such that $\delta_{p+1}=\delta_{p+2}=\cdots=\delta_{p+q}=1$ and $b_p=1$ simultaneously, for some $p\geqq 0$, then we can partition G as follows:

* One first triangularizes the matrix and then subtriangularizes it.

† H. W. Turnbull and A. C. Aitken, Theory of Canonical Matrices, London 1950, p. 68.

§ Inspired by Turnbull and Aitken.

$$G = \begin{pmatrix} D & 0 & 0 \\ E & N & 0 \\ F & 0 & M \end{pmatrix},$$

where D is $(p+1)\times(p+1)$ and N is $(q+1)\times(q+1)$. We note that E is zero except in its upper-left corner and that:

$$G^i = \begin{pmatrix} D^i & 0 & 0 \\ N^{i-1}E & N^i & 0 \\ M^{i-1}F & 0 & M^i \end{pmatrix},$$

G^{q+2} has a zero first column.

Let

$$\Omega = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & & & & & \cdot \\ \cdot & 1 & 0 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix},$$

and let e_i be the ith column of the k-rowed identity matrix. Then define

$$R = (e_2, e_3, \cdots, e_{p+1}, e_1, Ge_1, G^2e_1, \cdots, G^{q+1}e_1, e_{p+q+3}, \cdots, e_k) = \begin{pmatrix} \Omega & Q & 0 \\ 0 & I & 0 \\ 0 & B & I \end{pmatrix},$$

where the I's are identity matrices of the appropriate dimensions, $q+1$ and $k-p-q-2$, respectively. Since

(a)
$$R^{-1} = \begin{pmatrix} \Omega^{-1} & -\Omega^{-1}Q & 0 \\ 0 & I & 0 \\ 0 & -B & I \end{pmatrix},$$

(b)
$$\Omega^{-1}(D - QE)\Omega = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \delta_1 & 0 & & & & & \cdot \\ \cdot & \delta_2 & 0 & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & \delta_{p-1} & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \delta_p & 0 \end{pmatrix},$$

and

$$\text{(c)} \qquad F - BE = 0, \qquad MB - BN = 0, \qquad DQ - QN = 0,$$

we have

$$R^{-1}GR = \begin{pmatrix} \Omega^{-1}(D - QE)\Omega & 0 & 0 \\ E\Omega & N & 0 \\ 0 & 0 & M \end{pmatrix},$$

which is in classical form.

2. Uniqueness. If $T=\text{diag}\,(T_1, T_2, \cdots)$ and $S=\text{diag}\,(S_1, S_2, \cdots)$ are two reductions of A to classical form with T_i and S_i having the same root λ_i, then there exists a nonsingular $P=(P_{ij})$ such that $PT=SP$, hence $P_{ij}T_j=S_iP_{ij}$ for all i, j. But if $i\neq j$, it is easily shown by recursion starting from the last column of P_{ij} that $P_{ij}=0$. Hence T_i is similar to S_i for all i.

Now let T be any T_i and S the corresponding S_i. Then we may write $T=\lambda I+N$, $S=\lambda I+M$. The similarity of T and S implies that N^r and M^r have the same rank for all r and hence the same number of ones (on the subdiagonals). All chains of successive ones of length $r-1$ in N will disappear at exactly the rth power of N, and similarly for M. Also, the change in rank from N^r to N^{r+1} equals the number of chains of length $\geqq r>0$ in N, and similarly for M. Since N^r and M^r have the same rank, N and M must have the same number of chains of length r for each r.

THE TOEPLITZ-SILVERMAN THEOREM AND THE DEFINITION OF e^x

M. I. AISSEN, Johns Hopkins University

The discussion of the limit of $(1+x/n)^n$ as $n\to\infty$ through integer values is often reduced to the study of the limit of $(1+1/y)^{yx}$ as $y\to\infty$ through arbitrary values. This involves the notion of arbitrary exponents. An alternative method of defining e^x is with the power series $\sum_0^\infty x^n/n!$ whose convergence can be established very early in most discussions of sequences and series. Even before the discussion of the exponential function, the student should be able to understand the statement and proof of the Toeplitz-Silverman theorem, in at least the simplest case when the matrix is nonnegative and triangular. This note is to show how this theorem can be used to establish the existence and value of $\lim_{n\to\infty}(1+x/n)^n$ in terms of the known convergent series $\sum_0^\infty x^n/n!$

We first state the Toeplitz-Silverman theorem in the simple form and then apply it.

(TS) THEOREM. *Let $p_{n,k}$ be a double sequence of nonnegative real numbers satisfying $\sum_{k=1}^n p_{n,k}=1$, $n=1, 2, \cdots$, and $\lim_{n\to\infty} p_{n,k}=0$, $k=1, 2, \cdots$. If S_n is a sequence of complex numbers which converges to a number S, and if T_n is defined by $T_n=\sum_{k=1}^n p_{n,k}S_k$, then $\lim_{n\to\infty} T_n=S$.*

For our application, we define $S_n(x)=\sum_{k=0}^{n} x^k/k!$, and $T_n(x)=(1+x/n)^n$, for $n=1, 2, \cdots$. Then

$$T_n(x) = \sum_{k=1}^{n} p_{n,k}S_k(x), \tag{1}$$

where

$$p_{n,k} = \frac{k(n!)}{n^{k+1}(n-k)!}. \tag{2}$$

(In the classroom, (1) and (2) should be derived and not merely stated.)

Since $T_n(0)=S_m(0)=1$, we have $\sum_{k=1}^{n} p_{n,k}=1$ and from (2) it is clear that $\lim_{n\to\infty} p_{n,k}=0$. Hence by (TS) we have proved:

If $\sum_{k=0}^{\infty} x^k/k!$ *converges, then the sequence* $(1+x/n)^n$ *converges to* $\sum_{k=0}^{\infty} x^k/k!$. *It is clear that the result is immediately valid for complex* x.

The importance of the Toeplitz-Silverman theorem in its more general form and of its integral analogues certainly justify the inclusion of some form of it in the curriculum soon after the notion of "limit of sequence" is introduced. This particular application might provide a reasonable place for it.

PARTICULAR SOLUTIONS FOR NONHOMOGENEOUS, LINEAR, ORDINARY DIFFERENCE EQUATIONS

LOUIS C. BARRETT AND FORREST DRISTY, South Dakota School of Mines and Technology

In his classroom note* G. E. Latta explains how Lagrange's identity may be utilized to determine particular integrals of nonhomogeneous, linear, ordinary differential equations. The present note contains a parallel treatment of nonhomogeneous, linear, ordinary difference equations.

Let the nth order linear difference operator L be defined by

$$Ly(k) \equiv p_0(k)y(k+n) + p_1(k)y(k+n-1) + \cdots + p_n(k)y(k). \tag{1}$$

Applying the formula

$$V(k)U(k+r) \equiv \Delta[V(k-1)U(k+r-1) + V(k-2)U(k+r-2) + \cdots + V(k-r)U(k)] + V(k-r)U(k)$$

to each term in the right hand member of

$$v(k)Ly(k) \equiv v(k)p_0(k)y(k+n) + v(k)p_1(k)y(k+n-1) + \cdots + v(k)p_n(k)y(k), \tag{2}$$

we obtain the identity

$$v(k)Ly(k) \equiv \Delta P[y(k), v(k)] + y(k)\bar{L}v(k), \tag{3}$$

* Particular integrals for nonhomogeneous, linear, ordinary differential equations, this MONTHLY, vol. 65, 1959, pp. 624–625.

where

$$P[y(k), v(k)] \tag{4}$$

is a bilinear form, linear in $y(k)$, $y(k+1), \cdots, y(k+n-1)$ and in $v(k-n)$, $v(k-n+1), \cdots, v(k-1)$, and where

$$\bar{L}v(k) \equiv p_0(k-n)v(k-n) + p_1(k-n+1)v(k-n+1) + \cdots + p_n(k)v(k). \tag{5}$$

From (3) it is evident that any solution of

$$\bar{L}v(k) = 0 \tag{6}$$

makes $v(k)Ly(k)$ an exact difference. We shall call $\bar{L}$ and L adjoint difference operators and say that (6) and $Ly(k)=0$ are adjoint difference equations of one another. Equation (3) and expression (4) are, respectively, the counterparts of Lagrange's identity and the bilinear concomitant of differential equation theory.

Now, suppose we are required to find a particular solution of the nonhomogeneous difference equation

$$Ly(k) = f(k). \tag{7}$$

This may be accomplished if n linearly independent solutions

$$v_i(k), \qquad i = 1, \cdots, n, \tag{8}$$

of (6) are known, and these may always be found by use of (3) if the general solution of $Ly(k)=0$ is known.

Assuming, then, that functions (8) have been found, we substitute $Ly(k) = f(k)$ and $\bar{L}v_i(k)=0$ into (3) and subsequently take indefinite sums over k. In this way we obtain the system

$$P[y(k), v_i(k)] = \sum_k v_i(k)f(k), \qquad i = 1, \cdots, n, \tag{9}$$

of n linear algebraic equations in the n unknowns $y(k), y(k+1), \cdots, y(k+n-1)$. The determinant of the coefficients of these unknowns reduces to a nonzero multiple of

$$\begin{vmatrix} v_1(k-n) & v_2(k-n) & \cdots & v_n(k-n) \\ v_1(k-n+1) & v_2(k-n+1) & \cdots & v_n(k-n+1) \\ \cdot & \cdot & \cdots & \cdot \\ v_1(k-1) & v_2(k-1) & \cdots & v_n(k-1) \end{vmatrix} \tag{10}$$

and this determinant, called Casorati's determinant, does not vanish because the functions (8) were supposed linearly independent. Hence, a particular solution of (7) may be found by solving system (9) for $y(k)$. This solution may be specialized by attributing definite values to the arbitrary constants of summation which are implicit in (9).

Example. For the equation

$$y(k+2) - y(k) = \ln (k+1)(k+2), \tag{11}$$

(3) becomes

$$v(k)[y(k+2)-y(k)]=\Delta[v(k-1)y(k+1)+v(k-2)y(k)]+y(k)[v(k-2)-v(k)]$$

and two linearly independent solutions of $v(k-2)-v(k)=0$ are $v_1(k)=1$ and $v_2(k)=(-1)^k$. Hence, equations (9) may, in this case, be particularized to

$$y(k+1)+y(k)=\sum_{n=1}^{k}\ln(n)(n+1),$$

$$(-1)^{k+1}y(k+1)+(-1)^k y(k)=\sum_{n=1}^{k}(-1)^{n+1}\ln(n)(n+1).$$

Solving for $y(k)$, we find that a particular solution of (11) is

$$y(k)=\tfrac{1}{2}\sum_{n=1}^{k}[(-1)^{n-k+1}+1]\ln(n)(n+1),$$

which simplifies to $y(k)=\ln(k!)$.

MATHEMATICAL EDUCATION NOTES

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THE NATIONAL CONTEST IN HIGH SCHOOL MATHEMATICS IN UPPER NEW YORK STATE

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This paper is a brief report of the experience of the Upstate Section of the Upper New York State Section of the Mathematical Association of America which first in 1958 administered the national contests sponsored by the M.A.A. and the Society of Actuaries. In this section, which includes all of New York State north of latitude 42 degrees N except for five western counties, industries over the entire area have cooperated in the program by making available cash awards that have been given in the names of those industries to top-ranking winners.

Answers have been recorded on IBM sheets. An especially designed IBM sheet was used in the 1959 contest. All answer sheets have been returned to contest headquarters at the State University of New York College of Education at Albany where the scoring and the checking of the scoring has been performed by machine under operation by qualified student help.

Scholarships have been made available by those colleges and universities in the state that have cooperated in the program. In 1959 twenty colleges and universities made twenty-one scholarships available, and three institutions let it be known that they would give careful consideration to winners identified by this contest. At the time of the writing of this paper, it is known that seven of the cooperating colleges have awarded or offered fourteen scholarships to top-ranking winners and that two colleges from outside the state have awarded three sizable scholarships to those winners.

Cash awards have been made available by industries that have cooperated in the program. In 1959 sixteen one-hundred dollar cash awards were provided, five by the IBM Corporation, four by the New York Telephone Company, two by the National Commercial Bank and Trust Company, Albany, and one each by Allegheny Ludlum Steel Corporation, Bristol Laboratories, Inc., Gannett Newspapers, State Bank of Albany, and Sterling-Winthrop Research Institute.

The one-stage program on an IBM basis in the Upstate Section has been successful during these two years from the standpoint of both finance and interest. In 1958, 3,300 students from 211 schools participated; in 1959, 5,500 students from 328 schools participated. The number of cooperating colleges increased from 11 to 23 during that time; the number of cooperating industries from 5 to 8. During the past year each participating school was charged $5.00 plus 10 cents for each test.

The chairman of the contest committee in the Upstate Section has been conducting a continuing study of the students who ranked in the top one per cent of those participating in the 1958 contest. The twenty-three students who were seniors are now attending college. The average grade of these students, over all subjects, for the first semester, was B+, in spite of the fact that the highest score made by one of them in the 1958 contest was considerably below the limiting score for Honor Roll status. Of the ten students who were not seniors, nine were juniors and one was a freshman. The freshman was aged thirteen at the time of the contest and had already passed the chemistry and physics courses offered by his school. He again ranked in the top one per cent in the 1959 contest. In fact, six of the nine who were not seniors who again competed, ranked in the top one per cent in the 1959 contest. The most frequent choice of career expressed by the 1958 group is teaching, with engineering and research next in line; the most frequent choice of career expressed by those who ranked in the top one per cent in the 1959 contest from whom there has been a response, is engineering, with teaching next in line.

The coming of the new IBM machine 9902, an Electronic Test Scoring Punch Machine, due shortly to be on the market, will make possible the responding to 150 multiple choice, five choice, questions on the regular IBM card, the scanning of the card, the performing of the scoring, the punching of holes, and the recording of grades. This machine can make possible more efficient administering and grading of the M.A.A. tests.

MU ALPHA THETA

JOSEPHINE ANDREE, Norman, Oklahoma

With the help of many mathematicians, a gift from Pi Mu Epsilon, and sponsorship by the Mathematical Association of America, Mu Alpha Theta, an honorary organization for high school students of mathematics was launched in 1957. Student members receive a membership certificate and an announcement is sent to the local newspaper. Many club sponsors tell how hard their students work to achieve membership. Clubs are expected to meet every month and are helped by suggestions for meeting topics and references.

Mu Alpha Theta publishes *The Mathematical Log*, issued thrice a year and edited by Josephine Andree. It features articles on college level mathematics written for a high school background. C. D. Olds of San Jose State College is the mathematics editor. Every issue contains a set of challenging problems provided by V. E. Hoggatt also of San Jose State College. In the *Log* are also selections from letters from various chapters telling of their activities and news items concerning such things as the Mu Alpha Theta traveling library.

The traveling library is a service offered free to chapters anywhere. About eighty from Florida to Alaska are currently taking advantage of it. A chapter needs only to send a request to the librarian, Mr. George Hunt, to receive half a dozen books selected from outstanding new publications on such topics as the calculus, set theory, geometry, integrated college mathematics, puzzles, statistics, number theory, modern algebra. After six weeks each chapter mails its library on to another school. These books give a new outlook on mathematics to many students.

Mu Alpha Theta sponsors a series of regional meetings which are often addressed by members of the Association. Students have come as far as 300 miles to attend regional meetings in Oklahoma, Northern California, Southern California, Pennsylvania, Missouri, Maryland, and Illinois.

THE MASTER OF ARTS IN THE TEACHING OF MATHEMATICS

LYLE E. MEHLENBACHER, University of Detroit

The University of Detroit has announced a program leading to the degree of Master of Arts in the Teaching of Mathematics. This degree program has the objective of preparing better teachers of mathematics in both the high school and the elementary school. The program is available to teachers who do not have the mathematical background necessary for starting the traditional Master of Arts degree in mathematics but who have the capacity to start with a concentrated course in modern concepts of calculus and to continue through courses which will provide a broad and substantial foundation for teaching in the modern curriculum in mathematics. The requirements for admission to the program include a bachelor's degree from an accredited college with a minimum of a C+

average, a teacher's certificate, at least one year of college mathematics, and an interest in teaching mathematics.

Since the applicant for this degree is likely not to have studied calculus, or to have studied the "cook book" brand of calculus many years ago, he takes a sequence of two courses entitled "Modern Concepts of Calculus." These courses cover the fundamentals of analytical geometry and calculus as a unified system from a modern rigorous point of view.

Another special course required for the degree is either "Problems in Teaching High School Mathematics" or "Problems in Teaching Elementary School Mathematics," depending upon the particular field of interest of the applicant. These are subject matter courses covering the basic mathematical concepts needed in teaching the modern courses in mathematics.

The remaining courses in mathematics include: Introduction to Modern Algebra (2 semesters), Foundations, Fundamental Concepts of Geometry, and Theory of Functions of a Complex Variable In order to meet the needs of individual programs, alternate courses may be selected from: Theory of Numbers, Theory of Probability and Statistics, Advanced Calculus, Differential Equations, Advanced Modern Algebra, Theory of Functions of a Real Variable, Point Set Topology. The completion of the program requires twenty-four semester hours of mathematics and six hours in a cognate field selected from one of the four fields of chemistry, education, philosophy or physics.

This program was first announced in the Spring of 1959. By September 1959 we had processed and accepted 69 applicants. Many of the applicants are participants in the NSF Institutes, but several are "on their own." Courses in the program are available in late afternoon, after school hours, and in the summer.

SECOND THOUGHTS ON MODERNIZING THE CURRICULUM*

D. M. Merriell, University of California, Santa Barbara

There is a well-known principle in history that the heresy of yesterday becomes the orthodoxy of today. The turn of events in revising the mathematics curriculum is a good illustration. To those at the College of the University of Chicago a decade ago, the mathematics now referred to as "new" had a freshness made more piquant by the awareness that it was pedagogically different and controversial. Rudiments of symbolic logic, sets, relations and functions as sets of ordered pairs, mathematical structure, the definition of a group: all of these were capable of providing a challenge to put across to secondary students and of arousing some intellectual excitement which one could communicate. Today the same material, though not yet by any stretch of the imagination entrenched in either schools or colleges, threatens to become a new orthodoxy, destined to be

* The editors of this Department believe that there is a place in the Department for expressions of individual opinion on critical issues in mathematics education. Publication of such expressions are not to be interpreted as indication of agreement on the part of the editors of the Department.

shaken down to a teachable form as dry and dusty as the present day material which it would displace.

One wonders whether the major effort which is taking place is not misdirected. First, there is the assumption that the old curriculum is obsolete. This point is made large in order to sell the new article. In a consumer society in which built-in obsolescence is accepted as normal and the new is valued as the best, this has a great appeal. Second, there is the assumption that the obsoleteness of the present curriculum accounts for our lack of mathematical power and that the new article is going to set things right. How is one to account for the fact that European countries continue to turn out more competent mathematicians per capita under an old-fashioned curriculum? And what more conservative and traditional curriculum can one discover than that of the schools in the Soviet Union? Surely the differences are accounted for not by what is taught but by the thoroughness of the teaching and the possibility of getting through a much larger quantity of preparatory material. And here, the possibility of improvement depends on some very complex social questions. It is well-known that it is very difficult to change an entrenched curriculum. But the possibility of changing social attitudes and values seems so much more remote that one suspects that the curriculum question has been singled out in order to give the illusion of progress rather than attempting to strike at the real heart of the educational problem.

One of the noticeable features about the new programs is the emphasis on language and a new point of view. Great importance is attached to the notion of structure. At the same time, large portions of Euclidean plane and solid geometry are considered expendable. Perhaps one point is being lost in the shuffle. The high school course in plane geometry, even after it had been watered down to suit the needs of mass education, was the one course which gave the students a feeling for mathematical structure. Moreover, it contained in its palmier days some genuinely interesting theorems; and in the case of those propositions horribly referred to as "originals," it gave the better students opportunities to exercise ingenuity and to develop inventive power. The very least one should ask of a new program is that it should do as well as the old in these respects. But what interesting theorems does one find in modern secondary mathematics? The University of Chicago course was conceived as a secondary level course and was more extreme than most of its successors have been. Yet even it contained no significant theorems from either set theory, group theory, or symbolic logic although the course was organized about these subjects. If it is true that more preparation is necessary than can be given to high school students in order to reach significant theorems, the suitability of such subjects should be questioned. The fact is that there are parts of mathematics, Euclidean geometry being one such, in which one can present interesting theorems at the secondary level.

The issue being raised here is not whether the new mathematics is of value. All of it is material which a mathematics student should acquire at some point

in his training. The question is whether the introduction of this material at the secondary level does in fact fulfill the needs of secondary education. Does it arouse interest? Does it stimulate students to pursue mathematics further? Does it build an appreciation for ingenuity, creativity, and elegance? College teachers are well aware of the limited interest on the part of young students in axiomatics and rigor. The fact that axiomatization is an end product, a result of an urge for orderliness and organization rather than of the disorderly creative impulse, should have some bearing on the stage at which it is suitable to introduce it. Moreover, if the result of a course based on foundation material is to convey the impression that structure and axiomatics are the principal contributions of mathematics, this is almost as one-sided and biased a presentation as the impression now conveyed by an over-emphasis on formal manipulation. It is also likely to be as sterile in developing the inventiveness which is the essence of good mathematical training.

One of the principal objectives of groups now at work on the school mathematics curriculum is to increase the number of students who elect mathematics. Material is to be selected so as to attract students capable of further study and also to communicate some reasonably approximate notion of what a mathematician does. It is hard to see how the choice of foundation material satisfies these criteria. Since one of the primary occupations of a mathematician is to solve problems, it would seem more natural to build an attractive course around a small number of interesting problems. No doubt some attention would have to be given to language and formal skills but the problems would be selected so as to emphasize ideas and methods. Distasteful though it might be to a mathematics teacher, no attempt would be made to achieve mastery of the skills by the students.

Could such a course in "mathematics appreciation" really be a success? Selecting appropriate problems would be a difficult task but there is a certain amount of literature already available. In other fields such as art, music, and literature, complex technical problems are involved but it is still considered possible to select material which gives insight into the problems and methods of the artist and which can be grasped in some sense by secondary students. Perhaps the greatest obstacle in giving a successful course of this type is the teacher. There is no really satisfactory way to measure appreciation, and measurement is very strongly built into mathematics teachers.

Nevertheless, now that the machinery for curriculum revision has been set up and the times are propitious, one should be flexible enough to investigate alternatives. It would be foolhardy to believe that there is any single best curriculum. One should also be candid enough to express doubts, match up proposals against objectives, and continually criticize and evaluate what is being done. And finally, it should be remembered that curriculum is only one part and not even the most important part of the problem of education.

Traveling Science Demonstration Lecture Program*

The Traveling Science Demonstration Lecture Program has been sponsored for three successful years by the Oak Ridge Institute of Nuclear Studies in cooperation with the National Science Foundation, Atomic Energy Commission and others.

During 1959–60, this program is being expanded by the National Science Foundation to include four separate program centers which will serve particular regions of the United States. In addition to Oklahoma State University, other centers will be located at the University of Oregon, Michigan State University and the Oak Ridge Institute of Nuclear Studies. The Oklahoma State University center will serve the region composed of the states of Oklahoma, Texas, New Mexico, Colorado, Kansas, Missouri, Arkansas and Louisiana.

How the program operates. A specially trained and equipped traveling science teacher will be scheduled in a school system for approximately one week during the 1959–60 school year. Prior to the scheduled appearance, the traveling teacher will pre-visit the school and in conference with local teachers work out the program for that school and community.

A typical program for a school would include three or four scheduled appearances per day, Monday through Friday, including (1) a general high school assembly, (2) a teachers' meeting, (3) a science club program, (4) one or two civic group appearances, and (5) ten to fifteen lecture-demonstrations on science and mathematics to particular classes or special interest groups within the school.

The program for a given school and community is very flexible and can be designed to require a minimum number of disruptions or irregularities to the regular school schedule.

Oklahoma State University is now sponsoring a traveling science teacher in Oklahoma in cooperation with the Oak Ridge program and with financial support from the Frontiers of Science Foundation of Oklahoma.

Carnegie Institute of Technology—Mathematics Education Committee

Four years ago a committee was formed at Carnegie Institute of Technology to study the coordination of high school and university curricula in mathematics. A series of luncheon meetings was held with high school administrators and teachers, representing schools from which Carnegie draws a majority of its undergraduate student body, to survey mutual problems. The committee members have fulfilled many speaking engagements before student groups and teacher organizations and institutes, conducted extended series of seminars with several school districts, and established an annual competitive scholarship examination in mathematics for high school seniors.

Under the chairmanships of Professor John H. Neelley and Professor Allen F. Strehler the committee has established mutually beneficial relations with many schools and hopes to broaden the field of its activities in the future.

Proficiency Examinations for Teachers

The 1959 Annual Meeting of the Council on Cooperation in Teacher Education (CCTE) of the American Council on Education, held in Washington in October, gave further consideration to possibilities for proficiency examinations in teacher education. The interest of CCTE in proficiency examinations was first formally expressed in a resolution approved at their 1957 Annual Meeting. During 1958 a project providing for the use of proficiency examinations as one means of certification was established in West Virginia, and planning for the development of such a program in Wisconsin was started. Reports from these states were given at a special CCTE conference held in April,

* Quoted from *Traveling Science Teacher Program*, Oklahoma State University.

1959. For more detailed information on this subject write to Howard R. Boozer, Secretary-Treasurer, CCTE, 1785 Massachusetts Avenue, N.W., Washington 6, D.C.

MAA Visiting Lecturer Program to Secondary Schools 1959–60

The Mathematical Association of America is sponsoring for a second year a regional Visiting Lecturer Program to Secondary Schools. The program is made possible by a grant to the Association from the National Science Foundation. While the grant is somewhat larger this year, the program is being conducted again on a regional basis with the plan that schools served during the academic year 1958–59 will not be served during 1959–60. Nine regions have been designated and scheduling of the visits of the lecturers in the various regions is in charge of regional representatives. The regions for 1959–60 and the respective regional representatives are as follows:

Alabama-Arkansas	Houston T. Karnes, Louisiana State University, Baton Rouge
California	Roy Dubisch, University of California, Berkeley
Connecticut-Maine-Rhode Island-Vermont	W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts
Delaware-District of Columbia-Maryland-West Virginia	John A. Brown, University of Delaware, Newark
Idaho-Oregon-Washington	Harvey M. Gelder, Western Washington College of Education, Bellingham
Illinois	Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana
Kansas-Missouri-Nebraska	Russell N. Bradt, University of Kansas, Lawrence
Montana-North Dakota-South Dakota	Adrien L. Hess, Montana State College, Bozeman
North Carolina-South Carolina	Thomas D. Reynolds, Duke University, Durham, North Carolina

In general lecturers will make trips of short duration, and, in many cases, particularly in California and the Northwest, at short distances from the colleges at which the mathematicians are located. The New England region will be served by Professor Israel Rose of the University of Massachusetts, who is on a leave-of-absence during the second semester; and the North Carolina-South Carolina region will be served similarly on a leave-of-absence basis by Professor W. Norman Smith of the University of Wyoming. Other mathematicians who have agreed to give part-time to the program are as follows:

Idaho-Oregon-Washington: Bradford H. Arnold, Oregon State College; Wilfred E. Barnes, Washington State University; Z. W. Birnbaum, University of Washington; K. A. Bush, University of Idaho; Donald W. Bushaw, Washington State College; J. Richard Byrne, Portland State College; Theodore S. Chihara, Seattle University; Robert E. Gaskell, Oregon State College; George R. Johnson, Idaho State College; Anthony E. LaBarre, Jr., University of Idaho; Calvin T. Long, Washington State University; Richard D. Mayer, Idaho State College; Albert Nijenhuis, University of Washington; Thurman S. Peterson, Portland State College; Robert W. Rempfer, Portland State College; Sheldon T. Rio, Western Washington College; Joseph B. Roberts, Reed College; Hans Sagan, University of Idaho.

California: T. M. Apostol, California Institute of Technology; William Bade, University of California; Richard A. Dean, California Institute of Technology; Harley Flanders, University of California; Bernard Friedman, University of California; Charles J. A. Halberg, Jr., University of California at Riverside; Vincent C. Harris, San Diego State College; Leon Henkin, University of California; Calvin V. Holmes, San Diego State College; Paul J. Kelly, University of California at Santa Barbara; Donald H. Potts, Long Beach State College; Murray Protter, University of Cali-

fornia; Raymond M. Redheffer, University of California at Los Angeles; Judson Sanderson, Jr., University of Redlands; Elmer B. Tolsted, Pomona College.

Other regions: Richard D. Anderson, Louisiana State University; William E. Briggs, University of Colorado; Burton H. Colvin, Boeing Scientific Laboratories; J. A. Cooley, University of Tennessee; C. W. Curtis, University of Wisconsin; Mary Dolciani, Hunter College; J. C. Eaves, University of Kentucky; Wade Ellis, Oberlin College; W. T. Guy, University of Texas; Adrien Hess, Montana State College; Gerald B. Huff, University of Georgia; B. W. Jones, University of Colorado; H. S. Kaltenborn, Memphis State College; W. G. Lister, New York State College on Long Island; Robert McKelvey, University of Colorado; Herman Meyer, University of Chicago; C. O. Oakley, Haverford College; W. V. Parker, Alabama Polytechnic Institute; D. P. Richardson, University of Arkansas; Lawrence A. Ringenberg, Eastern Illinois University; P. C. Rosenbloom, University of Minnesota; A. H. Wallace, Indiana University; Eldon Whitesitt, Montana State College; Fred Young, Montana State University; J. H. Zant, Oklahoma State University.

The lecturers will be prepared not only to give lectures on mathematical topics, but to confer with students and faculty singly and in groups. They will be glad to advise students on future opportunities for study and employment, to discuss teaching problems and curriculum with members of the staff, and to throw what light they can on practices at comparable institutions. In short, the lecturers will cooperate with the schools in all possible ways toward the furtherance of the aims of the program. It is expected that, in general, the visiting lecturers will spend one day in a school. In the larger cities the lecturers may plan to spend several days visiting a number of schools both in the city and in neighboring communities. On occasion it may be desirable for teachers of a number of schools to meet jointly with the lecturer, and it may be possible for the students, or a select group of students, of several schools to meet together at one school to hear the lecturer. The experience of last year indicates that in general lecturers should not address assemblies of all high school students.

Application forms for high schools will be sent upon request addressed to the regional representative who is arranging the itinerary for the region in which the school is located.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1396. *Proposed by* (1) *Dustan Everman, Arizona State College,* (2) *A. E. Danese, Union College,* (3) *K. Venkannayah, Government College, Mercara, India*

For the Fibonacci sequence $\{f_n\}$, where $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n\geqq 3$, show that

(1) every fifth number in the sequence is divisible by 5.

(2) $f_{n+h}f_{n+k}-f_nf_{n+h+k}=(-1)^nf_hf_k$. (All the identities in Ganis, *Notes on the Fibonacci sequence,* this MONTHLY, vol. 66, p. 129, are special cases of this.)

$$(3) \qquad H(f_1, \cdots, f_{2n-1}) = \begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_2 & f_3 & \cdots f_{n+1} \\ \cdot & \cdot & \cdots \cdot \\ f_n & f_{n+1} & \cdots f_{2n-1} \end{vmatrix} = 0, \qquad n \geqq 3.$$

E 1397. *Proposed by Leon Bankoff, Los Angeles, California*

Show that $\sum AI \leqq \sum AH$, where I is the incenter and H the orthocenter of a triangle ABC.

E 1398. *Proposed by A. N. Aheart, West Virginia State College*

If A, B, C are the angles of a triangle, show that $\cos A + \cos B + \cos C < 2$.

E 1399. *Proposed by H. S. Shapiro, New York University*

Counterfeit coins all weigh 9 grams and genuine coins all weigh 10 grams. One is given five coins of unknown composition, and an accurate scale (not a balance). How many weighings are needed to isolate the counterfeit coins?

E 1400. *Proposed by S. D. Pratico, Iona College, New Rochelle, N. Y.*

There are 6 red balls and 8 green balls in a bag. Five balls are drawn at random from the bag and placed in a red box; the remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?

SOLUTIONS

Two Related Triangles

E 1366 [1959, 423]. *Proposed by V. E. Hoggatt, Jr., San Jose State College*

Show that if a, b, c form a triangle, then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form a triangle.

I. *Solution by Chih-yi Wang, University of Minnesota.* Since

$$|\sqrt{b} - \sqrt{c}|\,(\sqrt{b} + \sqrt{c}) < (\sqrt{a})^2 < b + c < (\sqrt{b} + \sqrt{c})^2,$$

it follows that $|\sqrt{b} - \sqrt{c}| < \sqrt{a} < \sqrt{b} + \sqrt{c}$.

II. *Solution by the proposer.* A necessary and sufficient condition [see the Seventh Polish Mathematical Olympiad, Prob. 9, First Round (*The Mathematics Teacher*, vol. LI, no. 8, Dec. 1958, p. 587)] that a, b, c form a triangle is

$$a^2b^2 + b^2c^2 + c^2a^2 > (a^4 + b^4 + c^4)/2.$$

We must therefore show that $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ satisfy

$$ab + bc + ca > (a^2 + b^2 + c^2)/2.$$

This readily follows since $b + c > a$, $c + a > b$, $a + b > c$, whence $ab + ac > a^2$, $bc + ba > b^2$, $ca + cb > c^2$.

III. *Solution by R. T. Hood, Ohio University.* We shall show that if a, b, c form a degenerate or nondegenerate triangle, then $\sqrt[n]{a}$, $\sqrt[n]{b}$, $\sqrt[n]{c}$ (n a positive integer) form a nondegenerate triangle. Assume $\sqrt[n]{a}+\sqrt[n]{b} \leqq \sqrt[n]{c}$. Raising to the nth power and omitting $n-1$ terms of the expansion, we have $a+b<c$. Hence, if $a+b \geqq c$, then $\sqrt[n]{a}+\sqrt[n]{b} > \sqrt[n]{c}$. A cyclic permutation of the letters gives corresponding conclusions when $b+c \geqq a$ and when $c+a \geqq b$. The revised statement of the problem follows immediately.

IV. *Solution by J. L. Brown, Jr., Pennsylvania State University Ordnance Research Laboratory.* We prove the more general assertion that if a, b, c form a triangle, then $f(a)$, $f(b)$, $f(c)$ form a triangle, where $f(x)$ is any nonnegative, nondecreasing, subadditive function on the domain $x>0$.

From $a \leqq b+c$, we have $f(a) \leqq f(b+c) \leqq f(b)+f(c)$. Cyclic permutation of a, b, c then yields the additional inequalities $f(b) \leqq f(c)+f(a)$ and $f(c) \leqq f(a)+f(b)$; therefore $f(a), f(b), f(c)$ form a triangle. The solution to the stated problem then follows from the observation that $f(x)=x^{1/2}$ has the properties required for the above proof.

V. *Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y.* This property is a special case of Ex. 5, p. 26, in L. M. Blumenthal, *Theory and Applications of Distance Geometry*:

THEOREM T. *Let M be a metric space and $\phi(x)$ a monotone increasing concave (from below) function that vanishes for $x=0$. If distance pq in M be redefined as $\phi(pq)$, then the new space is metric.*

Proof. Let p, q, r be points of M so that $pq+qr \geqq pr$. Since ϕ is monotone, $\phi(pq+qr) \geqq \phi(pr)$. Since ϕ is concave,

$$\phi(pq) \geqq \{pq/(pq+qr)\}\phi(pq+qr), \qquad \phi(qr) \geqq \{qr/(pq+qr)\}\phi(pq+qr).$$

Adding we find $\phi(pq)+\phi(qr) \geqq \phi(pq+qr) \geqq \phi(pr)$. The other properties defining a metric are obvious. The hypotheses are satisfied for $\phi(x)=x^t$ if $0<t \leqq 1$.

Also solved by A. N. Aheart, Mansur Arbabi, Leon Bankoff, S. D. Beck, D. A. Breault, Mark Bridger, R. F. Brown and Joel Levy (jointly), J. H. Case, Leon Cohen, Frank Dean, Monte Dernham, C. W. Dodge, J. W. Ellis, H. B. Emerson, Irma Esrig, Hazel E. Evans, J. W. Fitzpatrick, Donald Garlock, George Glauberman, Sidney Glusman, Michael Goldberg, A. J. Goldman, L. D. Goldstone, Bernard Greenspan, Cornelius Groenewoud, Joseph Hammer, J. H. Hodges, Raymond Huck, J. Jordan, A. F. Kaupe, Jr., J. D. E. Konhauser, P. G. Kumpel, Jr., Morton Kuppreman, Leo Lapidus, D. C. B. Marsh, R. A. Melter, Morris Morduchow, L. H. Mitchell, D. L. Muench, A. A. Mullin, C. S. Ogilvy, Margaret Olmsted, D. J. Persico, C. F. Pinzka, Joseph Prieto and Max Rosenberg (jointly), B. E. Rhoades, L. A. Ringenberg, J. T. Rosenbaum, John Saccoman, C. M. Sandwick, Sr., Jack Silver, Benjamin Sims, W. B. Stovall, Jr., Anna Thompson and R. G. Thompson (jointly), Julius Vogel, T. C. Wales, Alan Wayne, R. C. Weger, and F. L. Wolf. Late solutions by R. J. Carmier, Olga D. Mitrinovic, Helge Pahus, L. J. Schneider, and E. L. Spitznagel, Jr.

An analogous problem is to show that if a, b, c, d, e, f are the edges of a tetrahedron, then $a^{1/n}$, $b^{1/n}$, $c^{1/n}$, $d^{1/n}$, $e^{1/n}$, $f^{1/n}$ ($n=1, 2, \cdots$) also form a tetrahedron.

Lapidus pointed out that essentially Theorem T quoted in Solution V is also found in G. Birkhoff, Metric foundations of Geometry, I, *Trans. Amer. Math. Soc.*, vol. 55, 1944, p. 46. An interesting question is: Are there other functions than those described in Theorem T that preserve the

metricity of the space? The answer is in the affirmative. As an example, Lapidus considered the space of points on the surface of a sphere, where the Euclidean (chord) distance between any two of the points is replaced by the shorter great circle arc through the points. If diametral points are identified, the resulting space is metric, but the transforming function is not one of those described in Theorem T.

Concerning the Alternating Harmonic Series

E 1367 [1959, 423]. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Prove that the numerator of the sum $1-1/2+1/3-\cdots$ to $[2p/3]$ terms is divisible by p when p is a prime greater than 3.

Solution by N. J. Fine, Institute for Advanced Study. The required sum is equal to

$$\sum_{1\le k<2p/3} 1/k - 2\sum_{1\le 2k<2p/3} 1/2k = \sum_{1\le k<2p/3} 1/k - \sum_{1\le k<p/3} 1/k$$
$$= \sum_{p/3<k<2p/3} 1/k = \sum_{p/3<k<p/2} 1/k + \sum_{p/3<k<p/2} 1/(p-k)$$
$$= \sum_{p/3<k<p/2} \{1/k + 1/(p-k)\} = p\sum_{p/3<k<p/2} 1/k(p-k).$$

Also solved by A. E. Danese, Joseph Hammer, D. C. B. Marsh, K. Venkannayah, and the proposer. Late solution by M. Sugunamma.

A Theorem about Quartics

E 1368 [1959, 423]. *Proposed by M. S. Klamkin, AVCO Research and Development*

Show that if all roots of $ax^4-bx^3+cx^2-x+1=0$ are positive, then $c\ge 80a+b$.

Solution by A. J. Goldman, National Bureau of Standards, Washington, D. C. The proof will show that 80 is the best possible constant.

(i) If $a=b=c=0$, the result is true.

(ii) If $a=b=0$, $c\neq 0$, then the roots r_1, r_2 of $cx^2-x+1=0$ are positive, so $1/c=r_1r_2>0$, hence $c>0$ and the result is true.

(iii) If $a=0$, $b\neq 0$, then the roots of $bx^3-cx^2+x-1=0$ are positive. Since $r_1r_2r_3=r_1r_2+r_2r_3+r_3r_1=1/b$, we have $1/r_1+1/r_2+1/r_3=1$, which (since all $r_i>0$) shows that all $r_i\ge 1$; hence $c/b=\sum r_i\ge 3$, so $c\ge 3b\ge b$ and the relation holds.

(iv) Suppose $a\neq 0$. Then

$$(c-b)/a = 1/2\sum_{i\neq j} r_ir_j - \sum_k r_k = \{(\sum r_i)^2 - \sum r_i^2\}/2 - \sum r_i.$$

Call this $f(r_1, r_2, r_3)$. We can use Lagrangian multipliers to determine the minimum of $f(r_1, r_2, r_3)$ in the open subset $r_1>0$, $r_2>0$, $r_3>0$, $r_4>0$ of the constraint set

$$\prod_{i=1}^{4} r_i = \sum_{j=1}^{4}\prod_{i\neq j} r_i \qquad (=1/a),$$

i.e., $\sum 1/r_i = 1$, since on this set the function is bounded below. The minimum occurs when all $r_i = 4$, $f_{\min} = 80$; that is, $(c-b)/a \geqq 80$ as desired.

Also solved by D. A. Breault, R. F. Brown and Joel Levy (jointly), Michael Goldberg, D. C. B. Marsh, Margaret Olmsted, C. M. Sandwick, Sr., Chih-yi Wang, and the proposer.

The nth Derivative of $y=(1+x^{1/2})^{2n+2}$ at $x=1$

E 1369 [1959, 423]. *Proposed by R. E. Shafer, University of California*

If $y=(1+x^{1/2})^{2n+2}$, show that $y^{(n)}=4(n+1)(n+1)!$ at $x=1$.

Solution by Chih-yi Wang, University of Minnesota. Define $z=(1-x^{1/2})^{2n+2}$. It is evident that $z^{(n)}]_{x=1}=0$. Hence

$$y^{(n)}]_{x=1} = y^{(n)} + z^{(n)}]_{x=1} = D^n\left(2\sum_{k=0}^{n+1}\binom{2n+2}{2k}x^k\right)\Big]_{x=1}$$

$$= 2(n+1)! + 2(n+1)(2n+1)n! = 4(n+1)(n+1)!.$$

Also solved by A. J. Goldman, J. Jordan, D. C. B. Marsh, Jack Silver, and the proposer.

Two Improper Integrals

E 1370 [1959, 424]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

(a) Evaluate $\int_0^\infty (1-e^{-x})^n x^{-n}dx$ for integral $n>1$.

(b) Evaluate $\int_0^\infty (1-e^{-x})^2 x^{-2}\cos^2 kx dx$ for arbitrary $k>0$.

Solution by David Zeitlin, Remington Rand UNIVAC, St. Paul, Minn. All page references below are to *An Introduction to the Theory of Infinite Series*, 2nd ed. revised, by T. J. Bromwich, Macmillan and Co., 1926.

(a) On p. 473 occurs the result:

$$\int_0^\infty \left(\sum_{r=0}^n A_r e^{-a_r x}\right)x^{-n}dx = \left[-1/(n-1)!\right]\sum_{r=0}^n A_r(-a_r)^{n-1}\ln a_r,$$

where $\sum_{r=0}^n A_r a_r^k = 0$, $k=0, 1, \cdots, n-1$. Since

$$(1-e^{-x})^n = \sum_{r=0}^n (-1)^r\binom{n}{r}e^{-rx},$$

we have $A_r=(-1)^r\binom{n}{r}$, $a_r=r$, and the conditions are satisfied. Thus

$$\int_0^\infty (1-e^{-x})^n x^{-n}dx = \left[(-1)^n/(n-1)!\right]\sum_{r=2}^n (-1)^n\binom{n}{r}r^{n-1}\ln r.$$

(b) Ex. 4 on p. 488 gives the result:

$$\int_0^\infty \sum_{k=1}^n a^{-a_k x}\{A_k(1+a_k x)+B_k x\}x^{-2}dx = -\sum_{k=1}^n B_k\ln a_k - \sum_{k=1}^n A_k a_k,$$

where $\sum_{k=1}^{n} A_k=0$, $\sum_{k=1}^{n} B_k=0$, and the real parts of a_k are positive or zero. Since $\cos^2 kx=1/2+(1/2)\cos(2kx)$, the desired integral, I, becomes

$$I = \ln 2 + \operatorname{Re}\left[\int_0^\infty (1-e^{-x})^2 e^{2ikx} x^{-2} dx\right] \Big/ 2.$$

We now apply the above result with $n=3$, $a_1=-2ki$, $a_2=1-2ki$, $a_3=2-2ki$, $A_1=A_3=1$, $A_2=-2$, $B_1=2ki$, $B_2=2-4ki$, $B_3=-2+2ki$. Thus

$$I = 2\ln 2 + k\arctan(k-1/4k) - \arctan k - \ln\left[(1+4k^2)/(1+k^2)\right]/2.$$

Also solved by George Glauberman, John Rainwater, W. F. Trench, Chih-yi Wang, and the proposer.

Alternative forms to part (b) are obtained by replacing the second term in the above answer by $-k$ arccot $(4k^3+3k)$ or by $2k$ arctan $2k-k$ arctan $k-k\pi/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4881. *Proposed by Burton Randol, The Rice Institute, Houston, Texas*

Is there a non-trivial entire function f, such that $\int_L f(z)dz$ vanishes for all straight lines L infinite in both directions? (This is suggested by Newman's problem no. 4721 [1957, 750].)

4882. *Proposed by I. S. Gâl, Cornell University*

An (ω, ∞)-filter in a set X is a filter $\mathfrak{F}$ in X which has the countable intersection property: $F_n \in \mathfrak{F}$ implies $\bigcap F_n \in \mathfrak{F}$ $(n=1, 2, \cdots)$. Prove that, if X is a metric space and if every (ω, ∞)-filter in X has a nonvoid adherence then the same holds for every open subspace of X.

4883. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If

$$F = 1 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n+}\,\frac{1}{a_n+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1},$$

and P_n/Q_n is the nth convergent of F, prove $\cot^{-1} Q_{2n} = \cot^{-1} P_{2n} + \cot^{-1} P_{2n+1}$.

4884. *Proposed by Hans Schneider, The Queen's University of Belfast*

It is known that a ring whose additive group is the group of rationals (mod 1) has trivial multiplication. (C. Chevalley, *Fundamental Concepts of Algebra*, Example on Chapter 3.) Find a nontrivial ring whose additive group is the group of reals (mod 1).

4885. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research, Wilmington, Mass.*

Determine the unique solution of the integral equation

$$F(x_1, x_2, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} F(y_1, y_2, \cdots y_n)dy_1dy_2\cdots dy_n.$$

(The uniqueness when $n=2$ was one of the problems in the 1958 Putnam competition.)

4886. *Proposed by Isaac Namioka, Cornell University*

Let X be a Hausdorff (T_2) topological space and let Y be its dense subset. Prove that $\kappa(X) \leqq 2^{2^{\kappa(Y)}}$, where $\kappa(A)$ denotes the cardinality of the set A. Show by an example that this inequality cannot be improved even when X is compact. Is this assertion true for T_1-spaces?

SOLUTIONS

An Invalid Inequality

4603 [1954, 571; 1956, 191]. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i=1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geqq\frac{n}{2}$$

equality occurring only if all denominators are equal.

Partial Solution by M. Herschorn and J. E. L. Peck, University of Alberta, Calgary. Let S_n represent the left member. Diananda has shown the inequality true for $n \leqq 6$ [*Extensions of an inequality of H. S. Shapiro*, this MONTHLY, vol. 66, 1959, 489–491]. A counter-example has already been given with $n=20$.

We show that the inequality is false for all even $n \geqq 14$. To do this we put

$$x_i = 1 + a_i\epsilon, \text{ for } i \text{ even}; \qquad x_i = a_i\epsilon,\ a_i \geqq 0 \text{ for } i \text{ odd}.$$

Upon expanding S_n in powers of ϵ we have $S_n = n/2 + A_n\epsilon^2 + O(\epsilon^3)$, where

$$A_n = \sum_{i=1}^{n/2}(a_{2i+1} + a_{2i+2} - a_{2i})(a_{2i+1} + a_{2i+2}) - a_{2i-1}(a_{2i} + a_{2i+1}).$$

For $n=14$ the vector $(0, 0, 0, 8, 3, 10, 7, 8, 10, 3, 10, -2, 5, -2)$ will make $A_{14} = -3$, so that for sufficiently small positive ϵ we will have $S_n < n/2 = 7$. The

vector also has three consecutive zeros. Now for any permissible vector $(a_1, \cdots, a_n)$ having three consecutive zeros we may find another in dimension $n+2$ having three consecutive zeros and for which $A_{n+2}=A_n$, thus completing an induction on the even integers starting at 14. To see this it is easiest to suppose, as we may from cyclic symmetry, that the last two coordinates of $(a_1, \cdots, a_n)$ are zero. Then in dimension $n+2$ we take the same first n coordinates and attach two more zeros. A simple calculation shows that $A_{n+2}=A_n$.

Considerations of continuity show that if there is a counter-example where some of the coordinates $a_1, \cdots, a_n$ are zero, then there is also a counter-example with coordinates not zero and thus all $x_i>0$. The cases $n=8$, 10, 12 are still unsettled, as is the case of odd $n>6$.

The above counter-example was found on the McGill University computer (IBM 650). Our thanks go to Richard V. Andree for suggesting the problem as a programming exercise.

Continuous Increasing Function

4765 [1957, 746; 1958, 637; 1959, 317]. *Proposed by J. L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let $f(x)$ be a real function defined for $x\geqq 0$. If (i) f has a finite upper bound in any finite interval, and (ii) there are two positive numbers h, k such that $x''-x'\geqq h$ implies $f(x'')-f(x')\geqq k$, then, given $\alpha>1$, there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-\alpha h)<f(x)<g(x)$ for all $x\geqq\alpha h$.

Solution by the proposer. Let $a=k/4$, $b=k/2h$,

$$g_1(x) = \sup\{a + f(\xi) + b(x-\xi) : 0 \leqq \xi \leqq x\}.$$

If $x'<x''$ we have

$$\begin{aligned} g_1(x') &= \sup\{a+f(\xi)+b(x'-\xi):0\leqq\xi\leqq x'\} \\ &= \sup\{a+f(\xi)+b(x''-\xi):0\leqq\xi\leqq x'\} - b(x''-x') \\ &\leqq \sup\{a+f(\xi)+b(x''-\xi):0\leqq\xi\leqq x''\} - b(x''-x') \\ &= g_1(x'') - b(x''-x'), \end{aligned}$$

so that g_1 is an increasing function; more precisely $g_1(x'')-g_1(x')\geqq b(x''-x')$. Obviously $g_1(x)\geqq a+f(x)>f(x)$.

Suppose now $x\geqq\alpha h$ and let $\xi(0\leqq\xi\leqq x-h)$ be so chosen that

$$g_1(x-h) < 2a + f(\xi) + b(x-h-\xi);$$

and let n be the greatest integer such that $\xi+nh\leqq x(n\geqq 1)$, whence

$$f(x) \geqq nk + f(\xi) > nk + g_1(x-h) - 2a - bnh \geqq g_1(x-h).$$

For any given integer N, we set $\alpha=1+N\epsilon$ and define the iterated mean value

$$g(x) = (\epsilon h)^{-N} \int_x^{x+\epsilon h} dt_N \cdots \int_{t_3}^{t_3+\epsilon h} dt_2 \int_{t_2}^{t_2+\epsilon h} g_1(t_1)dt_1,$$

which is obviously an increasing function with $N-1$ continuous derivatives. Since $x \leqq t_N \leqq \cdots \leqq t_2 \leqq t_1 \leqq t_2+\epsilon h \leqq \cdots \leqq x+N\epsilon h$, it follows that

$$g_1(x) \leqq g(x) \leqq g_1(x + N\epsilon h),$$

whence $g(x-\alpha h) \leqq g_1(x-\alpha h+N\epsilon h) = g_1(x-h) < f(x) < g_1(x) \leqq g(x)$.

Power Series with Binomial Coefficients

4835 [1959, 238]. *Proposed by F. H. Northover, Carleton University, Ottawa, Canada*

Prove

$$\sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu}^2 x^\nu = (1-x)^{1-2n} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 x^k.$$

I. *Solution by R. C. Read, University College of the West Indies.* It is sufficient to show that, for each ν, the coefficients of x^ν on the two sides of the proposed equation agree, *i.e.*, that

$$\binom{n+\nu-1}{\nu}^2 = \sum_{j=0}^{n-1} \binom{n-1}{j}^2 \binom{2n-2+\nu-j}{\nu-j}.$$

But this is already, with obvious change in notation, a known identity. For reference and a simple proof see Nanjundiah, *Remark on a note by P. Turán*, this MONTHLY, v. 65, 1958, p. 354.

II. *Solution by Immanuel Marx, Purdue University.* To prove the somewhat more general formula

$$\sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu}\binom{m+\nu-1}{\nu} x^\nu = (1-x)^{1-m-n} \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{m-1}{k} x^k,$$

one rewrites the coefficients in each of the sums in generalized factorial form and verifies immediately that each is a hypergeometric series, so that the problem becomes

$${}_2F_1[n, m; 1; x] = (1-x)^{1-m-n}\, {}_2F_1[-n+1, -m+1; 1; x].$$

This last is, however, a direct application of the known transformation formula

$${}_2F_1[a, b; c; z] = (1-z)^{c-a-b}\, {}_2F_1[c-a, c-b; c; z].$$

See, for instance, Magnus and Oberhettinger, *Special Functions*, p. 8, last line.

Also solved by L. Carlitz, H. W. Gould, Fritz Herzog, H. O. Pollok, W. F. Trench, C. C. Yalavigi, Chih-yi Wang, David Zeitlin, and the proposer.

A Property of Solutions of a Pell Equation

4836 [1959, 238]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the equation $x^2-dy^2=-1$ where d is a nonsquare integer, and suppose that $x=a$ and $y=b$ is any solution in integers. Show that there are infinitely many solutions (x, y) in which x is a multiple of a and y is a multiple of b.

I. *Solution by Georgia C. Smith, Spelman College, Atlanta, Ga.* It is easily seen that if (a, b) is a solution of the given equation, then all the number pairs (x, y) determined by $(a+b\sqrt{d})^n=x+y\sqrt{d}$, where n is odd, are also solutions. Evidently x, y are multiples of a, b, respectively.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* If $db^2=a^2+1$, then

$$d(4a^2+1)^2b^2=(4a^2+1)^2(a^2+1)=(4a^2+3)^2a^2+1.$$

Hence, if (a, b) is a solution of the given equation, then so also is $(4a^2+3)a$, $(4a^2+1)b$ a solution. The existence of one solution implies inductively that there are infinitely many of the type desired.

Also solved by W. J. Blundon, Jean M. Calloway, L. Carlitz, G. Di Antonio, Michael Goldberg, Peter Greiner, Fritz Herzog, Free Jamison, D. C. B. Marsh, E. J. F. Primrose, R. M. Warten, and C. C. Yalavigi.

A Further Property of Solutions of a Pell Equation

4837 [1959, 238]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the solutions (x, y) of the equation $x^2-dy^2=-1$. Show that every x is relatively prime to every y.

Solution by E. J. F. Primrose, The University, Leicester, England. It is well known that if a, b is the smallest solution of $x^2-dy^2=-1$, all (positive) solutions are given by x_r, y_r, where

$$x_r+y_r\sqrt{d}=(a+b\sqrt{d})^r, \qquad r \text{ odd}.$$

Now suppose x_r and y_s have a common factor λ. Then, if n is any odd multiple of r and s, x_n and y_n have the factor λ by Problem 4836. But this is clearly impossible since $x_n^2-dy_n^2=-1$.

Also solved by L. Carlitz, Fritz Herzog, and D. C. B. Marsh. Several readers interpreted the statement of the problem to require only that every x and its related y be relatively prime. The editor offers his apology for a statement that did not more definitely exclude this trivial interpretation.

An Integral Identity

4839 [1959, 239]. *Proposed by R. G. Bushman, University of Wichita*

Let $d(n)$ denote the number of divisors of n and $[x]$ denote the greatest integer $\leq x$. Determine the validity of the identity

$$\int_1^x \left(\frac{1}{v}\sum_{n=1}^{[v]} d(n)\right) dv = \int_1^x \left(\frac{[v]}{v}\left[\frac{x}{v}\right]\right) dv, \qquad x \geqq 1.$$

Solution by E. N. Gilbert, Bell Telephone Laboratories, Murray Hill, N. J. The sum $\sum_{n=1}^{[v]} d(n)$ may be replaced by $\sum_{D=1}^{[v]} [v/D]$ since each positive integer D divides exactly $[v/D]$ of the integers n in $1 \leqq n \leqq [v]$. The given left-hand integral is then

$$\int_1^x \sum_{D=1}^{[v]} [v/D]dv/v = \sum_{D=1}^{[x]} \int_D^x [v/D]dv/v = \sum_{D=1}^{(x)} \int_1^{x/D} [t]dt/t$$

$$= \int_1^x \sum_{D=1}^{[x/t]} [t]dt/t = \int_1^x [x/t][t]dt/t,$$

as required.

Also solved by L. Carlitz, N. J. Fine, Fritz Herzog, Robert Weinstock, and the proposer.

Editorial Note. In some of the other solutions both given integrals were reduced to one of the forms

$$\sum_{n \leqq x} d(n) \cdot \log (x/n), \qquad \sum_{rs \leqq x} \log (x/rs),$$

where, in the latter form, the sum is taken over all ordered pairs of positive integers (r, s) such that $rs \leqq x$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Games and Linear Programming. By S. Vajda. Wiley, New York, 1957. 106 pp. $1.75.

Each of the eleven chapters dispenses a bite-sized morsel. But in toto the reader has pleasantly digested the basic notions of game theory, linear programming, and their interconnections. The appetizer outlines the theory of games. The simplex method constitutes the *pièce de résistance*, with the inverse matrix method and the dual simplex method served as alternate entrees. Three times the seasoning and spices are liberally sprinkled, as geometrical interpretations (in two-dimensional situations) add savory flavor to the chewier algebraic development. In case the knife does not cut the meat at first, a chapter on possible complications in the simplex method reveals potential remedies for sharpening the blade. Dessert comes with duality. Then as we lean back to relax, our linear programming techniques solve our games for us. Finally as a sort of after-dinner coffee, the method of leading variables brings the repast to a close. Just 100

pages of text, and they are small—less than 120 square centimeters of material to be absorbed on a page. Unfortunately some cook misjudged the palatability of subscripts or the arrangement of them in the table decorations. Nevertheless, you should enjoy tasting this little monograph.

RICHARD A. GOOD
The University of Maryland

The Structure of Arithmetic and Algebra. By May Hickey Maria. Wiley, New York, 1958. xiv+294 pp. $5.90.

The publishers bill *The Structure of Arithmetic and Algebra* as "A clear, concise treatment of the fundamental principles of modern mathematics." The statement is incorrect in nearly all phases. The book consists of two parts which we shall review separately. The first twelve chapters contain a reasonably clear and precise—but certainly not concise—axiomatic presentation of the real numbers and careful proofs of their more important properties. The first chapter muses at length about mathematical proofs but not until the second chapter does any mathematics appear. In the next eleven chapters the real numbers are defined by seventeen axioms and some of their properties are discovered; but no concern is given to existence. Every detail of every proof is tediously referred to the appropriate definition, axiom, or pre-proved theorem in the best tradition of plane Euclidean geometry. Throughout the book the reader is faced with the unfortunate choice of memorizing one hundred and thirty-five idle, useless, and nearly meaningless abbreviations (such as TND which stands for "Theorem on Negative of a Difference") or skipping the whole thing. Since a proof is vitiated if the reader must look up every reference, the complete index of abbreviations is of use only for emergency aid. The problem of an adequate reference system in an elementary proof is eternal but it may be better solved orally by the teacher, rather than by a device in the textbook. The exercises demonstrate a further difficulty of the inordinate completeness of the text. Although the author has made a valiant effort, too often the interesting problem has been solved as a part of the text material and the exercises must be unnatural, trivial, or repetitious.

New ideas are introduced leisurely with adequate simple examples to aid the student to understand the principles involved. Too often, though, this degenerates into the philosophy that if an idea is worth stating once, it is better repeated five times.

The book is "modern" in spirit only in so far as it insists on rigor in algebra (and the demand for rigor is at least two thousand years old); it deliberately ignores the power of generalization. For example, in successive sections the set of residue classes of the integers modulo a prime number and the rationals are shown to satisfy the field postulates, but the parallelism of the systems is neither pointed out nor exploited.

The latter part of the book is loosely connected melange. The first of three

chapters concerns solutions of quadratic equations. Here again lack of generalization causes trouble. Only the real numbers have been considered so the discriminant of the equation must be arbitrarily restricted to be nonnegative.

The next chapter introduces decimal notation for the real numbers and some elementary ideas of limit, and leads the student toward the idea of countability and uncountability but never quite reaches them.

The last chapter has an over-simplified and misleading short history of number notations, a lengthy discussion of the abacus, and much detail on number bases.

While most elementary algebra books have a deplorable lack of rigor and detail of proof, and at best pay lip service to the axioms of the number system, the solution does not lie with the other extreme. It is interesting to see how the real numbers can be placed on a logical basis and derived from a few axioms. The real danger is that a student, having read such a text, would be well aware of what was involved in a careful proof and hate every tedious detail of it.

DONALD A. NORTON
University of California, Davis

Some Aspects of Analysis and Probability. Vol. 4, *Surveys in Applied Mathematics.* By I. Kaplansky, M. Hall, Jr., E. Hewitt, and R. Fortet. Wiley, New York, 1958. xi+243 pp. $9.00.

Here are four distinct survey articles bound together in one volume. They differ in content, and even more in style. Marshall Hall's article contrasts most sharply with the other three: in content, because "combinatorial analysis" is not "analysis" (as commonly understood in the trade), and in style, because the article is accessible to any mathematician, whatever his specialty. At the other extreme, Fortet writes exclusively for probabilists, showing no mercy toward amateurs. The articles by Kaplansky and Hewitt (which *are* related to each other in content) aim at a middlebrow audience.

Kaplansky begins his report on *Functional Analysis* by briefly "reviewing the progress that has been made on the problems posed by Banach." Many mathematicians will be grateful for the existence of a published account of the present state of these problems. He then gives an expert summary (in 22 pages!!) of recent work on locally convex spaces and their operators, Banach algebras, and group representations. In particular, the classification of operators is illuminated by comparison with standard equivalence relations for matrices. (The reviewer would have mentioned Grothendieck's doctoral dissertation in the paragraph about approximating completely continuous operators by operators of finite rank.)

Hewitt's *Survey of Harmonic Analysis* should be read together with the article by Mackey (Bull. Amer. Math. Soc., vol. 56, 1950, pp. 385–412). The Hewitt article gives a more or less complete catalog of theorems proved up to 1956

about function spaces defined over locally compact abelian groups. But Hewitt's point of view is uncompromisingly abstract, and he therefore has a right to ignore results that depend on special properties of (say) the reals or the integers.

In reading Fortet's *Recent Advances in Probability Theory* the reviewer found himself better qualified to appreciate the "recent advances" than the classical "probability theory" upon which these advances were based. (But on the other hand, it would be a bold mathematician who could claim expert knowledge of all four fields surveyed in this volume.) Fortet presents cogent arguments for the study of *Banach-valued* random variables, and sketches appropriate generalizations of various laws of large numbers, of the Glivenko-Cantelli theorem, and of the central limit theorem. In particular, an ordinary random variable depending on a discrete or continuous parameter can be looked at as a general random variable with values in some function space over the integers or the real line. Many current research problems deal with ordinary random variables that happen to arise as (nonlinear) functionals of Banach-valued random variables. In addition to the above abstract functional-analytic results, some of which are Fortet's and are apparently announced by him for the first time in this survey, there are two chapters dealing with estimates for the degree of approximation in central-limit-like theorems.

It is not impossible that Marshall Hall's admirable *Survey of Combinatorial Analysis* will have a permanent place among introductory expositions in this difficult, intriguing, and as yet largely uncivilized field. He has succeeded in presenting an impressive number of fundamental problems and methods under his three categories of Enumeration, Choice, and Block Designs. Many mathematical details appear in full. (The reviewer noticed, however, that Latin squares cropped up at least twice before they were finally defined on page 77. Of more moment, Pólya's name does not appear in any of the three bibliographies. Surely his Hauptsatz from Section 16 of *Kombinatorische Anzahlbestimmungen*, etc., Acta Mathematica vol. 68, deserves mention.)

In conclusion we should like to ask the publisher and authors (M. Hall again excepted) a nonmathematical question. Why publish these surveys in a book? They are intended as lively descriptions of current research, not as definitive historical monuments. The sooner they appear in print the better. But internal evidence strongly suggests that in fact there has been a time lag of more than two years. (Hewitt's article had to ignore recent work by Calderon and Herz in spectral synthesis; Kaplansky went to press before the survey in Uspehi 11 (1956) by Berezin, Gelfand, *et al.* of group representations; not to mention Malliavin, who last year proved spectral synthesis impossible for L^1 of any noncompact group.) Perhaps the best answer to our question is that existing research journals are, in fact, not very much interested in expository articles.

H. MIRKIL
Dartmouth College

Introduction to Logic and Sets. (Prelim. ed.) By Robert R. Christian. Ginn, Boston, 1958. vi+70 pp. $1.00.

This text was designed as a supplement to high school and college mathematics courses, toward enhancing facility among teachers and students in the language and ideas of sets. Although too tightly coherent for intermittent reference, it is very brief and can be covered entirely in addition to regular course work, without undue strain. The student receives more and better practice than usual at thinking and working with sets, and the book is to be recommended as quite valuable and workable at these levels. Very well chosen, developmentally ordered, and literally instructive exercise sets are a key factor to its "more leisurely, yet more systematic (treatment), than that in most texts."

(Part) I (sections) 3, 4, 5, 6 and *7 present broad portions of the propositional calculus. I.*8, *9 and *10 interestingly and informatively apply truth-functional analysis to "black box" (electronic) circuits and to switching networks. II.1 introduces "set"-notion and Venn diagrams; then II.2 presents "set-builders." II.3, 4, and 5 examine and compound singular "conditions" (denoted by one-variable open statements, and usually included in the propositional calculus) in terms of set-relationships; then II.6 embeds "conditions" in set-builders to erect the algebra of sets along traditional lines. In II.7, quantifiers (unusual ones; for instance, universal does not imply particular) develop in terms of sets, then are used to build the usual set-operators for products and sums of sets of subsets. Further quantificational material appears in II.8 and 9, and II.*10 and *11 close by sketching certain propositional inference-patterns.

The intensively set-grounded symbolism, manner, and order of exposition should delight the many teachers who are demanding all-out emphasis on sets. Comparative relationships of dependence, decidability, completeness, and consistency, however, all argue compellingly to others for the more traditional order, (1) propositional calculus, (2) quantification theory, (3) set theory. Fortunately, the choice is not exclusive, since this text's brevity leaves ample course time for alternative views.

Regrettably absent in this sort of text is the reduction of relations and functions between numbers, real-plane point-sets, and plane geometrical loci, to simple sets of ordered pairs of real numbers.

Final particulars: (1) Regarding symbolism, the more usual forms "$\{x \mid (x \in M) \wedge p(x)\}$", "$(\forall x)[(x \in M) \Rightarrow p(x)]$" and "$(\exists x)[(x \in M) \wedge p(x)]$" are as workable here as the text's versions of set-builder and quantifiers, but these expose much more crucial internal structure to view and manipulation. (2) Practice and terminology of quotation are absent and confused, respectively. (3) Of accidental but nontrivial errors, four in the text and twelve among the many exercises and answers are too few for serious complaint.

ROBERT L. STANLEY
Washington State University

Elementary Matrix Algebra. By Franz E. Hohn. Macmillan, New York, 1958. xi+305 pp. $7.50.

Without going too much into abstract algebra, the author succeeds in presenting a very excellent introduction into the subject. The treatment is, in general, suitable for a course at the junior-senior level. The chapter headings will give an adequate idea of the contents: I. Introduction to Matrix Algebra; II. Determinants; III. The Inverse of a Matrix; IV. Rank and Equivalence; V. Linear Equations and Linear Dependence; VI. Vector Spaces and Linear Transformations; VII. Unitary and Orthogonal Transformations; VIII. The Characteristic Equation of a Matrix; IX. Bilinear, Quadratic and Hermitian Forms. In addition there are three appendices on the $\sum$ and $\prod$ notations, complex numbers, and isomorphisms. Every chapter contains a wealth of exercises, ranging "from purely formal computations and extremely simple proofs to a few fairly difficult problems." Throughout the book determinants are unnecessarily stressed. In Chapter II, the "sweep-out process for evaluating determinants" is developed, in Chapter III, inversion of matrices and solution of equations by "synthetic elimination." This seems to be an unnecessary duplication of effort since both methods are just slight modifications of Gaussian elimination. The term "cancelling" (*e.g.*, p. 61) should not be used without some statement of the cancellation law in an integral domain.

The book is unusually free of errors and is written in an easily readable style.

ALBERT NEWHOUSE
University of Houston

Ordinary Differential Equations. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1958. xv+534 pp. $9.75.

This is an introduction to the theory of ordinary differential equations by an able and prolific writer of mathematical textbooks, *e.g.*, *Advanced Calculus*, and *A First Course in Functions of a Complex Variable.* The book is happily influenced by the branch of engineering known as systems analysis or instrumentation. The book covers: basic ideas, linear equations and systems of linear equations including the application of matrices, exact differential equations, an introduction to nonlinear equations, power series solutions, numerical methods, phase plane analysis, and a final, twelfth chapter on fundamental theory, *i.e.*, existence and uniqueness theorems. Every chapter concludes with suggested references which lead the reader to many of the genuinely important books on ordinary differential equations and to a representative selection of important books on their scientific and engineering application. The reader would do well to handle and scan each of the references. This bibliographic knowledge, in addition to the basic training provided by the text, will enable the serious creative engineering mathematician to cope effectively with a significant area of technical problems. The book, in detail, is well written, modern and alive.

NATHAN GRIER PARKE, III
Carlisle, Massachusetts

Mathematics and Wave Mechanics. By R. H. Atkin. Wiley, New York, 1957. xv+348 pp. $6.00.

The author's intention has been "to select a course of study which is locally connected and yet adequate for the pursuit of modern quantum theory." The contents of the book thus consist of fifteen chapters, the first nine of which supply preparatory mathematical and physical background, *e.g.*, on integration, differentiation, multiple integrals, special functions, vectors, determinants, matrices, classical mechanics, wave motion and the electromagnetic field. The remaining six chapters deal with selected topics in quantum mechanics including a chapter on quantum chemistry and another on the theory of a quantised field.

To this reviewer it seems that the book can very well succeed in preparing the student for the pursuit of quantum mechanics, but not for attaining an understanding or mastery of it to the extent of being able to do research in it. This criticism is directed primarily to the treatment of quantum mechanics which is mostly a collection of miscellaneous, unrelated examples with little pretense of providing a thorough discussion of the physical basis or logical structure of the subject.

Granting this shortcoming, which is compelled by the inclusion of so many topics, the book should have its main value as an aid to self study for students with deficient mathematical background. Here again, the emphasis is mainly on manipulative techniques to be applied in the solution of particular problems rather than an abstract or rigorous formulation of the mathematics. The level and intent of the book are perhaps best characterized by noting that the references are preponderantly to the Cambridge Tripos examinations.

D. L. FALKOFF
Brandeis University

Integral equations. By F. Smithies. New York: Cambridge University Press, 1958. x+172 pp. $4.50.

This little book is an interesting introduction to the theory of nonsingular integral equations. The material covered corresponds roughly to the chapter on integral equations in the *Methods of Mathematical Physics* by Courant and Hilbert. The treatment of the subject matter is somewhere between the classical method, with its emphasis on piecewise continuous functions, and the modern functional analysis approach. Most of the theory is for the case of $\mathfrak{L}^2$ kernels and wide use is made of real and complex analysis; on the other hand, the author does not use the theory of linear operators on topological vector spaces. The price and the reward of this restraint are of the usual kind: the proofs are harder, the unifying concepts missing, but the convergence theorems are stronger.

The following is a list of major topics covered by the author. In Chapter I the reader is introduced to the various kinds of integral equations and an assortment of facts concerning Lesbegue integration. However, if this book were

used as a text for a graduate course, students could not be expected to learn enough about integration from Chapter I, hence courses in real and complex variables would have to be prerequisites. Chapter II defines and studies the elementary properties of resolvent kernels, characteristic value problems, the Neumann series, and certain generalizations of the Neumann series. Chapters III, V, and VII all treat various aspects of the Fredholm Theory. The approach in Chapter III is to approximate arbitrary $\mathfrak{L}^2$ kernels by kernels of finite rank. The usual theorems about characteristic values are then obtained. Chapters V and VI give explicit formulas for the solutions in the form of determinants for the case of continuous and $\mathfrak{L}^2$ kernels respectively. Chapter IV is an introduction to the study of orthonormal systems of functions. Chapters VII and VIII deal with the theory of Hermitian kernels. This is an extension of the more classical study of symmetric kernels. For the case of Hermitian kernels it is possible to describe the totality of characteristic values. Various expansion theorems are possible in this case.

Finally it should be noted that this is an unusually readable book and hence quite suitable for a graduate text.

FELIX HAAS
Wayne State University

Gödel's Proof. By E. Nagel and J. R. Newman. New York University Press, New York, 1958. x+118 pp. $1.75 paper, $2.95 cloth.

This book concerns the famous Incompleteness Theorem of Gödel, published in 1931. It is for the nonspecialist, and is based on an article in *Scientific American* in 1956. It falls into three parts: mathematical-historical background; outline of the proof; and (very briefly) discussion of its implications for mathematics. The first part is excellent, the second good, and the third open to substantial objections.

The first part, Chapters I–V, occupies, with its related appendices, most of the book. It surveys development of the axiomatic method, consistency proofs, formalization in symbolic logic, and Hilbert's search for "finitistic" proof of the absolute consistency of formalized mathematics. It is mathematical intellectual history at its best, presented with subtlety, wit and care,—suggestive and profitable for both layman and specialist.

In the second part, Chapters VI–VII, the authors face the difficulty that they must omit much detail and must thus leave room for possible misunderstanding. An intelligent selection of key steps is made and summaries of the connecting arguments are given—on the whole, a very good job. The reviewer makes two suggestions. (1) In connection with the arithmetization of metamathematics, it should be pointed out that some but not all statements of the metalanguage can be made within arithmetic. The "solution" to Richard's Paradox lies not so much in the distinction between arithmetic and metalanguage as the authors allow us to believe, but rather in the further distinction between those statements of the metalanguage that can and cannot be repre-

sented within arithmetic. A reference to work of Tarski would have been appropriate here (and at some other places). (2) In connection with the "undecidable" statement G that asserts its own unprovability, the authors state that G is true. This is ambiguous and somewhat misleading. In the case of an application of Gödel's technique to a simple axiomatization of arithmetic it is indeed the case that we can infer the truth of G by a conventional (and quite precise) argument. But Gödel's technique can also be applied to a stronger system in which all the conventional methods of mathematical argument are included. Let G be the "undecidable" arithmetical statement now produced. We cannot infer that G is true. We know that G is true if the stronger system is consistent; but at best we can only say, on empirical or aesthetic grounds, that the consistency of this stronger system is probable.

In the brief third part, Chapter VIII, the authors first assert, correctly, that the theorem seems to show the impossibility of success in Hilbert's search. They then draw the (admittedly) more controversial conclusion that the mathematical powers of the human mind transcend (in some sense) mechanical procedures. This inference is not uncommon, and is, in the reviewer's opinion, unwarranted and probably wrong. It arises from the erroneous idea, implied in the text, that an iterated application of the Gödel method for making undecidable but true inferences is in some inherent way incapable of axiomatization. In fact, such iteration out through large segments of the transfinite ordinals can be included in the axioms. Of course any such inclusion can be extended by Gödel's methods, but this is true in the same fairly trivial sense that any effective system of notation for a segment of ordinals can be extended to cover a larger segment. The limitation on a machine is one of size, not quality. The revolutionary information conveyed by Gödel is not that the human mind transcends mechanical procedures, but that mathematics itself does so. The authors also fail to make clear (see (2) above) that such iteration could break down at any point with an initial appearance of inconsistency. Recent work on ordinal notations and degrees of unsolvability sheds much light on these matters.

Apart from these criticisms, the authors merit praise for an unusually good and most welcome piece of popular exposition.

HARTLEY ROGERS, JR.
Massachusetts Institute of Technology

Lineare Algebra. By Werner Graeub. Band XCVII, Die Grundlehren der Mathematische Wissenschaften. Springer-Verlag, Berlin-Göttinger-Heidelberg, 1958. xi+219 pp. 35.70 DM (About $8.55).

Beginning on page one with axioms for a linear space, first over the real field and then over an arbitrary (commutative) field, this book develops systematically in the customary style and format of this ("Grundlehren") series the standard basic theory of linear operators in the finite dimensional case. Proofs are usually phrased in invariant form, without reference to the matrix describing a mapping, but the notation is frequently highly explicit so that A is said to be

the determinant of the matrix (α_i^k), *cf.* p. 56. Tensors are introduced as multilinear functions of vectors in a pair of dual spaces and the interplay between the tensor descriptions and abstract vector space terminology should be informative to readers acquainted with only one of these notations.

The determinant is axiomatized as a function of n vectors with specified properties and is used in a treatment of orientation. Metric spaces receive much attention both in the definite and indefinite case and affine and euclidean classification of conics and quadrics is explicit.

The elementary divisor theory concludes the book. Rational and Jordan canonical forms are obtained by adaptation of bases to invariant spaces.

A slightly unusual item is the proof (attributed to John Hilnor) by highly nonalgebraic methods that two real quadratic forms in $n>2$ variables may simultaneously be diagonalized if they do not both vanish for the same nonzero vector.

No errors were noted and the exposition is exemplary. It may appear strange to some readers that dual mappings are described by matrices (α_ν^μ) and (β_ν^μ) with $\alpha_\nu^\mu=\beta_\nu^\mu$, yet the matrices are described (p. 40) as transposes of one another under the agreement that in one case ν is the row and in the other the column index. The tensor notation thus hints at the correct conclusion that in a pair of dual spaces (best understood as a right and a left vector space over a division ring) the same matrix, and not a pair of transposed, most conveniently describes the dual mappings. (*Cf.* N. Jacobson, *Lectures in Abstract Algebra*, vol. 2, p. 58, a reference missing from the very brief bibliography.)

W. GIVENS
Wayne State University

Asymptotic Methods in Analysis. By N. G. deBruijn. Interscience, New York, 1958. xii+200 pp. $5.75.

This most entertaining and informative book is written for the nonexpert in the field and explains by examples some of the more important methods of asymptotic analysis. The mechanism of asymptotic series, the Lagrange inversion formula, the Laplace method for integrals, saddle point techniques, etc. is made plain by problems from differential equations, iterated functions, etc. The author's philosophy is that specific difficult problems, if worked out in detail, involve ideas which are general and which can be applied to general classes of problems, a view with which the reviewer concurs entirely.

MORRIS NEWMAN
National Bureau of Standards, Washington, D. C.

Algebra. By J. W. Archbold. Pitman & Sons, London, 1958. xix+440 pp. 45/-. (About $6.30).

As stated in the preface this text was written to include the algebra mentioned in the syllabuses for certain general and honors Bachelor's degrees in mathematics at the University of London.

The first three chapters contain properties of the number systems and finite series. This is followed by algebraic inequalities and then complex numbers. The next seven chapters are devoted to polynomials, polynomial equations, roots of unity, and symmetric functions. Chapters 15 and 21 are on determinants and Chapters 16 and 20 present some group theory. Chapters 17–19 and 22–24 deal with vector spaces, matrices, and quadratic forms, while the last chapter concerns discriminants and resultants.

The ordering of material could be improved by juxtaposing the two chapters on determinants and also the two on groups. Unnecessarily, vector spaces and matrices are restricted to fields of characteristic zero. Unfortunately only nonsingular linear transformations are mentioned and then only relative to quadratic forms. Most of their important properties are omitted.

In general the material is presented well. An outstanding feature is the set of exercises; some extend the theory and provoke thought while others are interesting problems that have appeared in the MONTHLY and the *Mathematical Gazette*.

This book will be useful as a reference. It is also worthy of consideration as a text in one or a sequence of undergraduate courses. For an advanced undergraduate course in modern algebra, however, a text which goes more deeply into groups, rings, fields, linear transformations, etc. and with a more abstract approach would be more suitable.

HOWARD E. CAMPBELL
Michigan State University

Mathematical Aspects of Subsonic and Transonic Gas Dynamics. By Lipman Bers. Wiley, New York, 1958. xv+164 pp. $7.75.

This extensive survey of the mathematical aspects of a specific field in gas dynamics, namely, theory of steady two-dimensional compressible flow, is a tremendous task. It includes most of the important information and results (for which the author is one of the principal contributors) from a vast source of some 400 papers published since World War II in a relatively compact book of 135 pages. The author's elucidating style and inspiring way of presenting the problems, the mathematical background and theory for their solution, the results and their extensions, and the open problems for future research make the book delightfully readable as well as valuable to applied mathematicians and aerodynamicists. The book is mainly devoted to two categories of problems: those of subsonic flows and those of transonic flows. The principal mathematical problem is, above all, the theorem of existence and uniqueness.

For subsonic flow theory, a detailed account of the mathematical background for elliptic partial differential equations is given, such as the smoothness of solutions, maximum principle, quasiconformal mapping, theory of pseudo-analytic functions, the direct method of calculus of variation and Schauder's fixed point theory. The theorem of existence and uniqueness for a flow, uniform at infinity, past an obstacle is precisely stated and the main feature and ideas

for the proof are clearly indicated and sketched. Of course, no details could be given in such a compact treatise. However, in each case, the sources are always referred to. The subsonic channel flow is treated by means of the extension of Riemann's mapping theorem. A brief account is also given for subsonic flows with free boundaries.

For the mathematical background of transonic flows, the author is mainly dealing with the uniqueness and existence of various boundary value problems of linear partial differential equations of mixed type. A special feature concerning the question of uniqueness is the presentation of yet unpublished work of Friedrichs on the new approach of the abc method as a systematic procedure for finding correctly set boundary-value problems for a wide class of linear partial differential equations. The question of "transonic controversy" (the nonexistence of transonic shockless flow past a profile) arouses a great interest and the author urges and foresees further research and progress.

Finally, the author considers transonic flows with free boundaries, in particular, the flow behind a detached shock wave due to a truncated wedge. In this particular occasion, the author did not make the important remark that the flow behind a curved shock wave is rotational. Consequently, the Chaplygin equation (3.4), as used by Frankl, is at most an approximation (by neglecting the vorticity). As a matter of fact, the exact equation involves an *a priori* unknown function (of the stream function), which is determined by the yet-to-be-found shape of the shock. [Sauer, *Gas Dynamics*, p. 137, (218); also, N.A.C.A. TN2364, p. 7, (12)]. It is indeed a new free boundary problem. A few trivial misprints are found, such as in (6.7), (8.4), (2.12), (13.9), and the lines (from the bottom) 7, p. 69; 4, p. 71; 12, p. 76; 4 and 5, p. 91.

S. S. Shu
Purdue University

NEWS AND NOTICES

Edited by Lloyd J. Montzingo, Jr., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Emeritus R. W. Brink, University of Minnesota, represented the Association at the inauguration of Dr. Harvey M. Rice as President of Macalester College.

Professor A. B. Cunningham, West Virginia University, represented the Association at the inauguration of Dr. Elvis Jacob Stahr, Jr. as President of West Virginia University on October 3, 1959.

Professor F. A. Ficken, New York University, represented the Association at the Centennial Celebration at Cooper Union on November 2, 1959.

Alabama Polytechnic Institute: Assistant Professor J. B. Crawford retired on June 15, 1959; Associate Professor Nathaniel Macon has been promoted to Professor of Mathematics and Director of the Computer Laboratory; Assistant Professor C. W. Huff has been promoted to Associate Professor; Assistant Professor Ben Fitzpatrick, Jr., University of Texas, Professor Margaret Baskervill, Shorter College, and Mr. J. C. Schoonmaker, New York University, have been appointed Assistant Professors; Miss Thelma J. Caraway, University of Arkansas, has been appointed Instructor.

University of California, Berkeley: Associate Professors S. P. Diliberto and Edmund Pinney have been promoted to Professors; Assistant Professors E. A. Bishop, H. O. Cordes, and Bertram Kostant have been promoted to Associate Professors; Professors S. S. Chern and E. H. Spanier, University of Chicago, have been appointed Professors; Dr. M. J. Greenberg, Rutgers University, has been appointed Instructor; Associate Professor J. W. Addison, Jr., University of Michigan, and Dr. Yitzhak Katznelson have been appointed Lecturers; Professors R. J. Godement, University of Paris, France, A. N. Milgram, University of Minnesota, and John Milnor, Princeton University, have been appointed Visiting Professors; Drs. Wanda Szmielew, University of Warsaw, Poland, and D. S. Carter have been appointed Visiting Associate Professors; Drs. A. A. Kosinski, University of Warsaw, Poland, and Azriel Levy have been appointed Visiting Assistant Professors; Dr. Jerzy Los, University of Warsaw, Poland, has been appointed Visiting Research Mathematician; Professor Hans Lewy will be on leave 1959–60 at New York University; Professors C. B. Morrey, Jr. and M. H. Protter are on leave to accept appointments with the Miller Institute for Basic Research; Associate Professor Henry Helson will be on leave in the Spring 1960 and will spend the term at the Institute for Advanced Study.

University of California, Santa Barbara: Drs. A. M. Bruckner and G. J. Culler, University of California, Los Angeles, and J. C. Ceder, University of Washington, have been appointed Assistant Professors.

University of Cincinnati: Associate Professor Arno Jaeger has been promoted to Professor; Visiting Research Professor A. J. Macintyre has been appointed Research Professor; Assistant Professor H. D. Lipsich has been promoted to Associate Professor and is on leave as an N.S.F. Faculty Fellow at Stanford University; Dr. Harry Ferguson, Wright Air Base, Dayton, Ohio, has been appointed Associate Professor; Dr. Sheila S. Macintyre, University of Aberdeen, Scotland, has been appointed Visiting Associate Professor; Assistant Professor K. T. Leung, Miami University, has been appointed Assistant Professor; Messrs. Cecil Craig, University of Kentucky, Covington, Kentucky, K. D. Parakh, Trinity School, London, England, Arnold Polak, Centralia Junior College, Washington, and Ralph Fairchild have been appointed Instructors.

University of Delaware: Associate Professor Russell Remage is on an N.S.F. Faculty Fellowship in Cambridge, England; Dr. W. V. Caldwell, University of Michigan, has been appointed Assistant Professor; Colonel D. N. Sundt, U. S. Army, Retired, has been appointed Lecturer.

De Paul University: Assistant Professors G. L. Weiss and W. F. Darsow have been promoted to Associate Professors; Drs. J. Z. Yav, Richard O'Niel, and Robert Szczarba, University of Chicago, have been appointed Instructors.

Drake University: Professor A. L. O'Toole, Western Illinois University, has been appointed Professor; Mr. Donald Schuette, University of Michigan, has been appointed Assistant Professor.

Florida State University: Associate Professor R. L. Plunkett has been promoted to Professor; Assistant Professor P. J. McCarthy has been promoted to Associate Professor; Professor M. L. Curtis, University of Georgia, has been appointed Professor; Dr. B. J.

Boyer, Purdue University, has been appointed Assistant Professor; Miss Valma Y. Edwards and Messrs. J. R. Neergard and W. O. Williford have been appointed Instructors.

University of Kansas: Associate Professor W. R. Scott has been promoted to Professor and Acting Chairman of the Department of Mathematics; Assistant Professor J. C. Lillo is on leave at the Institute for Advanced Study; Associate Professor U. W. Hochstrasser is on leave as Scientific Attache of the Swiss Embassy, Washington, D. C.; Dr. C. J. Himmelberg, Midwest Research, has been appointed Assistant Professor; Dr. A. P. Robertson, University of Glasgow, Scotland, has been appointed Research Associate; Mrs. Wendy Robertson has been appointed Visiting Assistant Professor; Mr. P. E. Pulley, Jr. has been appointed Instructor and Assistant Director of the Computation Center; Mrs. Shirley L. Deeter has been appointed Instructor.

Lebanon Valley College: Professor B. H. Bissinger has been installed to the John Evans Lehman Chair of Mathematics; Assistant Professor Robert Wagner is on leave and will spend the year at the University of Pennsylvania; Mr. Paul Henning has been appointed Assistant Professor.

University of Maine: Professor W. S. Lucas has retired with the title Professor Emeritus; Adjunct Professor L. H. Swinford, Brooklyn Polytechnical Institute, has been appointed Lecturer; Assistant Professor J. W. Toole, St. Peter's College, has been appointed Assistant Professor.

University of Maryland: Assistant Professor J. W. Brace has been promoted to Associate Professor; Mr. W. L. Jones, Queens College, has been appointed Instructor.

University of Minnesota: Assistant Professors L. W. Green and W. D. Munro have been promoted to Associate Professors; Dr. N. G. Meyers has been promoted to Assistant Professor; Drs. Helmut Röhrl, University of Chicago and the University of Munich, Germany, Avner Friedman, University of California at Berkeley, and J. A. Nitsche, University of Freiburg, Germany, have been appointed Visiting Associate Professors.

Mississippi State University: Mr. T. W. Daniel has been promoted to Assistant Professor; Miss Dora E. Kearney, University of Maryland, has been appointed Assistant Professor; Mrs. Marcelle T. McDonald, University of Chicago, and Miss Helen M. McPherson and Mr. T. R. Slough have been appointed Instructors.

University of New Hampshire: Assistant Professors S. L. Ross and R. H. Owens have been promoted to Associate Professors; Mr. D. M. Burton, University of Rochester, has been appointed Assistant Professor.

New York University: Professor Lipman Bers has been appointed Head of the Graduate Mathematics Department; Professor Morris Kline has been appointed Chairman of the Washington Square College Mathematics Department, succeeding Professor F. W. John who has retired with the title Professor Emeritus; Professor F. A. Ficken has been appointed Chairman of the University Heights Mathematics Department, succeeding Professor H. A. Giddings who has been put in charge of the N.Y.U. Advanced Engineering Training Program at the Bell Telephone Laboratories; Professor Andre Weil, Institute for Advanced Study, has been appointed Visiting Professor; Assistant Professor Allan Birnbaum, Columbia University, and Mr. H. B. Keller have been appointed Associate Professors; Associate Professor M. D. Davis, Rensselaer Polytechnic Institute, Hartford Graduate Center, has been appointed Visiting Associate Professor; Mr. S. S. Gupta, Bell Telephone Laboratories, has been appointed Adjunct Associate Professor; Mr. M. A. Kervaire, Battelle Memorial Institute, Geneva, Switzerland, has been appointed Assistant Professor; Mr. B. R. Levy has been appointed Instructor; Associate Professors K. S. Miller, J. T. Schwartz, and G. A. Yanosik have been promoted to Professors; Assistant Professor Jerome Berkowitz has been promoted to Associate Professor; Messrs. Martin Schechter, N. R. Stanley, and Peter Ungar have been promoted to Assistant Professors.

University of North Dakota: Mr. J. S. Rue has been promoted to Assistant Professor; Dr. E. O. Nelson has rejoined the staff as Associate Professor; Mr. L. E. Mauland has been appointed Instructor.

Northwestern University: Associate Professors E. H. C. Hildebrandt, and Hsien-Chung Wang have been promoted to Professors; Miss Helen M. Clark and Mr. Eben Matlis have been promoted to Assistant Professors; Assistant Professor S. G. Ghurye, University of Chicago, has been appointed Associate Professor; Dr. A. B. Simon, Yale University, has been appointed Assistant Professor; Dr. S. M. Shah, Muslim University, India, has been appointed Visiting Professor; Dr. Martin Eisen, University of Toronto, has been appointed Instructor.

Oberlin College: Assistant Professor John Baum has been promoted to Associate Professor and will be on leave at Princeton University on an N.S.F. Fellowship; Assistant Professor Robert Weinstock, on leave from the University of Notre Dame, has been appointed Associate Professor for the year 1959–60.

Oklahoma State University: Associate Professor F. A. Graybill has been promoted to Professor; Assistant Professor R. L. Caskey has been promoted to Associate Professor; Associate Professor D. R. Shreve has been promoted to Director of the Computing Center; Drs. R. W. Gibson, Kansas State Highway Commission, and D. L. Weeks have been appointed Assistant Professors.

University of Ottawa: Dr. Aubert Daigneault has been appointed Assistant Professor; Mr. C. J. Murphy has been appointed Lecturer.

University of Pennsylvania: Dr. Pincus Schub has been promoted to Assistant Professor and has received a $1000 award for excellence in undergraduate teaching; Dr. P. A. Caris has received a $1000 award for excellence in undergraduate teaching; Assistant Professor Smbat Abian, Queens College, and Dr. Alan Wilson, Rice Institute, have been appointed Assistant Professors.

University of Rochester: Assistant Professor R. A. Raimi has been promoted to Associate Professor and Acting Chairman of the Department of Mathematics; Professor J. F. Randolph has resigned as Chairman but will continue as Fayerweather Professor of Mathematics; Mr. N. J. Rothman has been promoted to Assistant Professor; Assistant Professor J. H. Case, University of Utah, and Dr. R. P. Goblirsch, University of Virginia, have been appointed Assistant Professors; Dr. S. K. Kaul, Hampton Institute, and Mr. F. A. Puff, West High School, Rochester, New York, have been appointed Instructors; Professor Dov Tamari, Haifa Technion and Jerusalem Hebrew University, has been appointed Visiting Professor.

Rutgers, The State University: Associate Professors L. H. Bunyan, R. M. Cohn, and V. L. Shapiro have been promoted to Professors; Assistant Professors B. H. McCandless and R. F. Gabriel have been promoted to Associate Professors; Drs. P. E. Martin, E. R. Gentile, and Terence Butler have been promoted to Assistant Professors; Professor C. A. Nelson, Douglass College, has retired; Dr. R. L. Swain, State Teachers College, New Paltz, New York, has been appointed Professor; Drs. J. C. E. Dekker, University of Kansas, Jerome Neuwirth, Massachusetts Institute of Technology, and E. J. Taft, Columbia University, have been appointed Assistant Professors; Drs. R. E. Bryan, Yale University, Stevens Heckscher, Harvard University, and Mr. T. S. MacHenry, New York University, have been appointed Instructors.

University of Saskatchewan: Assistant Professor Norman Shklov has been promoted to Associate Professor; Dr. P. M. Cuttle, University of Oregon, has been appointed Instructor.

South Dakota School of Mines and Technology: Associate Professor L. C. Barrett has been promoted to Professor; Assistant Professor C. A. Grimm has been promoted to Associate Professor.

University of Tennessee: Assistant Professor John Neuberger, Illinois Institute of

Technology, has been appointed Assistant Professor; Messrs. V. R. Hatz, Christian Brothers College, C. H. Edwards, and B. G. Wilson have been appointed Instructors.

U. S. Air Force Academy: Captain E. D. Wilmoth received Doctor of Science degree from Massachusetts Institute of Technology, June, 1959; Captains R. L. Eisenman, J. C. Gall, J. C. Marsh, E. R. Prince, Jr., R. S. Slizeski, and P. V. Vegna have been promoted from Instructors to Assistant Professors; Captains Edward Anlian, University of Iowa, F. F. Betzer, Sandia Base, New Mexico, R. R. Erbschloe, Harlingen Air Force Base, Texas, B. D. Harrison and J. H. Sullivan, North Carolina State College, B. C. Healy, Holloman Air Force Base, New Mexico, W. R. Lambert and M. C. Patton, University of Colorado, R. W. Oesch, Wright-Patterson Air Force Base, W. P. Rollins, University of Illinois, B. J. Welch, A. & M. College of Texas, Majors N. H. Smith, University of Colorado, and R. K. Moorhead, Scott Air Force Base, Illinois, and First Lieutenant N. P. Callas, University of Colorado, have been appointed Instructors.

Washington University: Professor T. L. Downs is on leave at Stanford University on an N.S.F. Faculty Fellowship; Associate Professor J. A. Jenkins, University of Notre Dame, has been appointed Professor; Assistant Professor W. M. Boothby, Northwestern University, has been appointed Associate Professor; Dr. Helmut Schiek, University of Bonn, Germany, has been appointed Visiting Professor.

Wellesley College: Dr. A. L. Gropen and Mr. R. J. Crittenden have been appointed Instructors.

Western Michigan University: Associate Professor Pearl Ford has retired with the title Associate Professor Emeritus; Mr. R. E. Sechler, Michigan State University, and Mrs. Adelaide E. Howson have been appointed Instructors.

Western Washington College of Education: Dr. S. T. Rio, Oregon State College, and Mr. W. J. Sanders, Chico State College, have been appointed Assistant Professors.

Wisconsin State College: Mr. C. E. Flanagan, Chairman of the Mathematics Department, has taken a year's leave of absence to continue his graduate study at the University of Wisconsin; Assistant Professor C. J. Vanderlin is Acting Chairman of the Mathematics Department; Mr. J. J. Dodd, University of Illinois, Mr. Hsuan Wei, Northwestern University, and Miss Alberta Christen, University of Wisconsin, have been appointed to the staff.

University of Wisconsin—Milwaukee: Assistant Professor Daniel Waterman, Purdue University, and Dr. R. J. Mihalek, Illinois Institute of Technology, have been appointed Assistant Professors; Mr. Robert Kurtz, Princeton University, has been appointed Instructor; Dr. Wojciech Slowikowski, Mathematical Institute of the Polish Academy of Sciences, has been appointed Visiting Lecturer.

Assistant Professor Asger Aaboe, Tufts University, has been promoted to Associate Professor.

Mr. Joseph Altinger, S. M., Cathedral Latin School, Cleveland, Ohio, has been appointed Teacher of Algebra at St. James High School, Chester, Pennsylvania.

Mr. W. E. Anderson, South Dakota State College, has accepted a position as Research Scientist with Aero Research, Honeywell Regulator Company, Minneapolis, Minnesota.

Mr. M. T. Austin, Chrysler Missile Division, Detroit, Michigan, has accepted a position as Senior Engineer with the Curtiss-Wright Corporation, Wood-Ridge, New Jersey.

Mrs. Helen G. Bass, Van Antwerp School, Schenectady, New York, has been appointed Teacher at West Rocks Junior High School, Norwalk, Connecticut.

Mr. A. I. Benson, General Electric Company, Tempe, Arizona, has accepted a position as Technical Advisor on SAGE with the System Development Corporation, Santa Monica, California.

Associate Professor Gerald Berman, Illinois Institute of Technology, has been appointed Professor at the University of Waterloo, Ontario, Canada.

Professor Dorothy L. Bernstein, University of Rochester, has been appointed Professor at Goucher College.

Dr. J. R. Boen, University of Illinois, has been appointed Assistant Professor at Illinois Southern University.

Mr. O. L. Buchanan, Jr., University of Kansas, has been appointed Mathematics Teacher at Sandy Springs High School, Sandy Springs, Georgia.

Assistant Professor A. T. Butson, University of Florida, has accepted a position as Research Specialist with the Boeing Airplane Company, Seattle, Washington.

Professor Emeritus C. C. Camp, University of Nebraska, has been appointed Lecturer at the Dearborn Center of the University of Michigan.

Visiting Associate Professor Buchanan Cargal, University of Texas and Chulalongkorn University, Bangkok, has accepted a position as Member of the Technical Staff of Land-Air, Inc., Pacific Missile Range, Point Mugu, California.

Captain G. C. Carlstedt, Bradley University, has been appointed Instructor at California State Polytechnic College.

Dr. A. F. Cayford, University of California, Los Angeles, has been appointed Instructor at the University of British Columbia.

Dr. R. E. Collins, Humble Oil and Refining Company, Houston, Texas, has been appointed Associate Professor of Physics at the University of Houston.

Assistant Professor W. R. Cowell, Montana State University, has accepted a position as a Member of the Technical Staff of Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. D. W. Crowe, University of Michigan, has been appointed Lecturer at University College, Ibadan, Nigeria.

Associate Professor R. B. Davis, on leave from Syracuse University, is a Visiting Lecturer at Yale University during 1959–60.

Dr. D. W. Dean, University of Rochester, has been appointed Instructor at Yale University.

Dr. H. E. Debrunner, Princeton University, has been appointed Research Instructor at Duke University.

Associate Professor F. C. DeSua, College of William and Mary, has been appointed Associate Professor at Northern Illinois University.

Dr. R. S. DeZur has been appointed Assistant Professor at the University of Wyoming.

Professor J. E. Dotterer, Manchester College, has retired with the title Professor Emeritus.

Mrs. Ellen H. Dunlap, Wayne State University, has been appointed Assistant Professor at Northeastern University.

Miss Ruth A. Fish, Contra Costa Junior College, Richmond, California, has been appointed Instructor at Foothill College.

Mr. R. B. Foster, Jr., University of New Mexico, has accepted a position as Staff Member of the Sandia Corporation, Albuquerque, New Mexico.

Dr. Walter Gautschi, National Bureau of Standards, has accepted a position as Mathematician with the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Mr. K. G. Getman, John Wiley and Sons, Inc., has accepted a position as Science and Mathematics Editor with Henry Holt and Company, New York.

Mr. Martin Goldsworth, East Carolina College, has been appointed Assistant Professor at Louisiana Polytechnic Institute.

Mr. R. M. Gordon, Burroughs Corporation, has been appointed Associate Director of the Computation Center at Stanford University.

Assistant Professor F. C. Hall, Manhattan College, has been appointed Assistant Professor at St. John's University, Brooklyn, New York.

Mr. D. E. Harris, Jr., Los Alamos Scientific Laboratory, has accepted a position as Senior Mathematician with Chance Vought Aircraft, Inc., Dallas, Texas.

Mr. W. F. Herrin, State University of Iowa, has been appointed Instructor of Physics at North Park College.

Dr. Robert Heyneman, Cornell University, has been promoted to Assistant Professor.

Dr. H. E. Hoffman, University of Oklahoma, has accepted a position as Physicist with the Office of Naval Research, Fluid Dynamics Branch, Washington, D. C.

Assistant Professor V. E. Hoggatt, Jr., San Jose State College, has been promoted to Associate Professor.

Mr. Henry Hosek, Jr., Ball State Teachers College, has been appointed Instructor at Purdue University Calumet Center, Hammond, Indiana.

Mr. W. A. Husted, Hobart College, has accepted a position as Senior Insurance Analyst for the Metropolitan Life Insurance Company, New York.

Mr. F. J. Kammel, Georgetown University, has been appointed Instructor at Loyola College, Baltimore, Maryland.

Assistant Professor C. E. Kerr, Lafayette College, has been appointed Assistant Professor at Dickinson College.

Dr. H. C. Kranzer, New York University, has been appointed Associate Professor at Adelphi College.

Associate Professor Joachim Lambek, on leave from McGill University, is a Member of the Institute for Advanced Study.

Rev. C. J. Lewis, S.J., on leave from Fordham University, is on an N.S.F. Science Faculty Fellowship at Harvard University.

Miss Hildegarde Liebscher, Drake University, has been appointed Instructor at Western Reserve University.

Mr. C. A. Long, University of Illinois, has been appointed Instructor at Bowling Green State University.

Mr. R. M. Lutz, University of Cincinnati, has been appointed Instructor at Geneva College.

Mr. T. H. MacGregor, Rutgers, The State University, has been appointed Assistant Professor at Lafayette College.

Dr. J. E. Mack, Purdue University, has been appointed Assistant Professor at Ohio University.

Dr. N. F. G. Martin, Iowa State University, has been appointed Instructor at the University of Virginia.

Assistant Professor L. F. McAuley, University of Wisconsin, has been appointed Visiting Associate Professor at Louisiana State University for 1959–60.

Mr. E. H. McKinney, University of Pittsburgh, has been appointed Assistant Professor at Northern Illinois University.

Mr. L. D. McLean, Jr., University of Arizona, has been appointed Instructor at Hofstra College.

Mr. E. G. McNiel, DePaul University, has accepted a position as Mathematical Analyst Senior with the Lockheed Aircraft Corporation, Sunnyvale, California.

Mr. N. L. Menzie, University of Nebraska, has accepted a position as Associate Engineer with the Boeing Airplane Company, Wichita, Kansas.

Mr. J. K. Moore, Massachusetts Institute of Technology Instrumentation Laboratory, has accepted a position as Computer Programmer with the Space Technology Laboratories, Los Angeles, California.

Dr. J. M. Moser, St. Louis University, has been appointed Assistant Professor at San Diego State College.

Mrs. Sarah B. Murray, Johns Hopkins University, has accepted a position as Applied

Science Representative for International Business Machines Corporation, Denver, Colorado.

Dr. Amin Muwafi, Ministry of Education, Amman, Jordan, has been appointed Assistant Professor at the American University, Beirut, Lebanon.

Assistant Professor Eliezer Naddor, Johns Hopkins University, Baltimore, Maryland, has been promoted to Associate Professor of Industrial Engineering.

Mrs. Lois J. Niemann has been appointed Instructor at Colorado State University.

Mr. J. J. O'Neill, Pure Oil Company, Chicago, Illinois, has accepted a position as Staff Analyst with the Datamatic Division of Minneapolis Honeywell, Newton Highlands, Massachusetts.

Dr. C. C. Oursler, Indiana University, has been appointed Assistant Professor at Southern Illinois University, East St. Louis, Illinois.

Assistant Professor W. D. Peeples, Jr., Alabama Polytechnic Institute, has been appointed Professor at Howard College.

Assistant Professor E. E. Posey, West Virginia University, has been appointed Associate Professor at Virginia Polytechnic Institute.

Dr. Anthony Ralston, Bell Telephone Laboratories, Whippany, New Jersey, has been appointed Lecturer at the University of Leeds, England.

Dr. D. L. Richter, University of North Carolina, has been appointed Assistant Professor in the Department of Statistics, University of Minnesota.

Mr. L. L. Ross, Babcock and Wilcox Company, Barberton, Ohio, has accepted a position as 709 Applied Programmer with the International Business Machines Corporation, Poughkeepsie, New York.

Professor K. M. Saksena, Government College, Nainital, India, has been appointed Professor at the Indian Aid Mission, Embassy of India, Kathmandu, Nepal.

Associate Professor O. P. Sanders, Louisiana Polytechnic Institute, has been appointed Head of the Mathematics Department and Director of Engineering Studies at Hardin-Simmons University.

Mr. R. P. Savage, University of Tennessee, has been appointed Assistant Professor at Tennessee Wesleyan College.

Professor Peter Scherk, University of Saskatchewan, has been appointed Professor at the University of Toronto.

Mr. Anthony Sepan, University of Wisconsin, has been appointed Assistant Professor at Coe College.

Dr. Norman Severo, National Bureau of Standards, has been appointed Associate Professor and Acting Head of the Department of Statistics at the University of Buffalo.

Assistant Professor R. F. Shortt, South Dakota School of Mines and Technology, has been appointed Assistant Professor at San Fernando State College.

Professor Jack Silber has returned to Roosevelt University from a leave of absence as Consultant to the Operations Analysis Office, Hq. United States Air Forces in Europe.

Sister M. Peter Stengl, Holy Family College, Manitowoc, Wisconsin, has been appointed Principal of Xavier High School, Appleton, Wisconsin.

Associate Professor W. E. Stuermann, University of Tulsa, has been promoted to Professor of Philosophy.

Mr. Leonard Sweet, Akron Public Schools, has been appointed Assistant Professor at the University of Akron.

Assistant Professor J. B. Wells, Jr., University of Louisville, has been appointed Assistant Professor at the University of Kentucky.

Mr. B. G. Willis, White Sands Proving Ground, Las Cruces, New Mexico, has accepted a position as Mathematician with the Lockheed Aircraft Company, Sunnyvale, California.

Mr. R. J. Winkelman, University of Rochester, has accepted a position as Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Assistant Professor Dale Woods, Idaho State College, has been appointed Associate Professor at Northeast Missouri State Teachers College.

Mr. R. E. Wyllys, Defense Department, Washington, D. C., has accepted a position as Associate with the Planning Research Corporation, Los Angeles, California.

Mr. A. W. Yonda, AVCO, Lawrence, Massachusetts, has accepted a position as Senior Member, Technical Staff, Radio Corporation of America, New York.

Mr. H. B. Zarian, United States Army, has accepted a position as Mathematician with Shell Development, Emeryville, California.

Dr. P. W. Zehna, Stanford University, has been appointed Assistant Professor at Colorado State College.

Miss Ruth B. Eddy, D. C. Heath and Company, died on July 29, 1959. She was a member of the Association for ten years.

Professor Emeritus V. S. Mallory, Montclair State College, New Jersey, died September 30, 1959. He was a member of the Association for forty-two years.

A 1960 International Congress

An International Congress for Logic, Methodology and Philosophy of Science will be held at Stanford University, Stanford, California, U.S.A., from August 24 to September 2, 1960, under the auspices of the International Union for History and Philosophy of Science.

The proceedings of the Congress will be organized into the following eleven sections:

1. Mathematical logic.
2. Foundations of mathematical theories.
3. Philosophy of logic and mathematics.
4. General problems of methodology and philosophy of science.
5. Foundations of probability and induction.
6. Methodology and philosophy of physical sciences.
7. Methodology and philosophy of biological and psychological sciences.
8. Methodology and philosophy of social sciences.
9. Methodology and philosophy of linguistics.
10. Methodology and philosophy of historical sciences.
11. History of logic, methodology and philosophy of sciences.

The proceedings will consist of a number of invited addresses, in addition to brief contributed papers. The closing date for contributed papers is March 1, 1960. All questions concerning the Congress should be addressed to Professor Patrick Suppes, Serra House, Stanford University, Stanford, California, U.S.A.

Notice

The Division of Mathematics, National Academy of Sciences—National Research Council, announces the appearance of its annual list of Visiting Foreign Mathematicians. This bulletin includes information regarding mathematicians and statisticians spending some part of this academic year in the United States and gives the dates of their visit together with the host institution. Copies may be obtained by writing to the Division of Mathematics, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, Washington 25, D. C.

Handbook for Automatic Computation

Preparation of a handbook for automatic computation, in five or more volumes, is now under way. It will appear in the series *Grundlehren der Mathematischen Wissenschaften*. Dr. A. S. Householder, Oak Ridge National Laboratory, is working with six European editors in the preparation of this handbook.

The purpose of the handbook is to provide a collection of tested algorithms for mathematical computations of all sorts. These algorithms are to be written in ALGOL, hence will be usable on any machine for which a suitable translator is available, and even without a translator can be used as a model for programming. The descriptive language will be English. Each algorithm is to be accompanied by enough explanatory information to make it understandable, along with whatever information is available on speed, accuracy, range, or, more generally, for judging the effectiveness of the algorithm for a given type of problem. In any event, only pretested algorithms will be published.

Contributions are earnestly solicited. For the present, at least, these must necessarily be in the form of actual algorithms, along with information as to the extent and mode of testing the algorithm, estimates of accuracy, and experience in using it. As algorithms are published, information relating to published algorithms also will be welcomed. Contributions may be sent to the editor named above.

New SIAM Lecture

The Society for Industrial and Applied Mathematics has announced The John von Neumann Lecture which will be presented annually at a national meeting of the Society. The first lecture will be presented at the summer meeting in 1960. The lecturer will survey and evaluate a significant field of pure mathematics in the light of its contribution to major advances in applied mathematics. An authoritative lecture presented from this point of view is considered to be in harmony with the research and philosophy of Professor von Neumann.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 333 persons have been elected to membership by the Board of Governors on applications duly certified.

Roger W. Adair, Student, Emory University.
Allen A. Adams, Student, Central State College, Oklahoma.
Sheldon Adelberg, A.B. (New York) 79 East 36 Street, Paterson 4, New Jersey.
Paul Akin, J.D. (Northwestern) Agency Manager, Federal National Mortgage Association.
Bernard S. Albert, M.B.A. (California) Senior Operations Research Analyst, North American Aviation.
Carol R. Alcorn, M.A. (Stanford) Instr., Bakersfield College.
Leroy R. Amunrud, M.S. (Montana S.U.) Instr., Montana State College.
Salvatore Anastasio, B.A. (Cathedral Coll.) Part-time Instr., Iona College; Grad. Student, New York University.
Charles N. Anderson, M.Eng. (North Carolina S.C.) Instr., North Carolina State College.
John M. Anderson, Ph.D. (California) Professor, Pennsylvania State University.
Ray D. Anderson, Jr., B.S. (Portland S.C.) Grad. Student, University of Idaho.
Walter G. Andrews, Student, St. Michael's College, Vermont.
Harry W. Appelgate, B.A. (Hofstra Coll.) Programmer, Service Bureau Corp.
Loren N. Argabright, M.A. (Kansas) Lecturer, Seattle University.
Fred Armstrong, Student, St. Michael's College, Vermont.
Hack Arroe, Ph.D. (Copenhagen) Head, Physics & Math. Div., University of Denver.
Thomas S. Bagby, Jr., Student, University of Kentucky.
Malvina M. Baltrukovicz, B.A. (Hunter Coll.) Asst. Actuary, New York City Retirement System.
C. Lamar Bannister, B.S. (Jacksonville S.C.) Mathematician, Army Ballistic Missile Agency.
Joseph Barback, B.A. (Buffalo) Grad. Student, Rutgers University.
Richard C. Basinger, Student, Missouri School of Mines.
Roques Bejarano, A.B. (California, Berkeley) Asso. Engineer, Jet Propulsion Lab.
David L. Bennett, B.Ed. (Alberta) Principal, Bashaw Schools, Alberta.
Paul W. Berg, Ph.D. (New York) Asso. Professor, Stanford University.
Mrs. Louise Berge, A.A. (California) Student, University of California, Berkeley.
Edward P. Berger, A.M. (Duke) Instr., University of Nevada.
Roy Bertoldo, B.S. (Queens Coll.) Student Actuary, New York Life Insurance Co.
Stuart M. Block, M.A. (New York) Instr., Hunter College High School.
Osei K. Bonsu, B.S. (Oregon S.C.) Grad. Student, Oregon State College.
Wilfred E. Boykin, A.B. (Emory) Grad. Student, Emory University.
Allen L. Bradfield, A.M. (Indiana) Asso. Professor, Vincennes University.
Warren Brisley, B.S. (Sydney) Lec-

turer, Newcastle University College.
Rev. Fintan A. Bromenshenkel, M.S. (Notre Dame) Instr., St. John's University, Minnesota
Brother Emeric, B.S. (St. John's, New York) Head of Dept., Xaverian High School, Brooklyn, New York.
Brother Theodose Brown, M.A. (Laval) Professor Academie de Quebec.
Lee Bruce, M.A. (Chicago) Teacher, Judson Junior High School, Salem, Oregon.
John R. Bruggeman, M.S. (Denver) Engineer, Thiokol Chemical Corp.
Irene E. Buckles, M.Ed. (Alberta) Teacher, Canadian Armed Forces, Europe.
Barry Bunow, Student, University of California, Berkeley.
Edward H. Burger, M.A. (Miami) Head of Dept., Pensacola Junior College.
Donald N. Butler, M.Ed. (Whittier Coll.) Teacher, LaPuente Union High School, California.
Lt. Nicholas P. Callas, M.A. (Colorado) Instr., United States Air Force Academy.
Michael L. Cantor, M.A. (Columbia) Teacher, Junior High School 263, Brooklyn, New York.
Charles P. Cassidy, B.A. (Assumption, Canada) Teacher, Sudbury Mining & Technical School, Ontario.
Edward F. Cavey, A.B. (Fordham) Teacher, Georgetown Preparatory School, Garrett Park, Maryland.
Ward O. Chase, M.A. (St. Lawrence) Head of Dept., Potsdam Central School, New York.
Don C. Cherrington, M.A. (Stanford) Head of Dept., Castlemont High School, Oakland, California.
Bruce M. Chetty, Student, Humboldt State College.
Lawrence W. Chinnery, Student, Grinnell College.
Hope H. Chipman, M.A. (Columbia) Instr., University High School, Ann Arbor, Michigan.
Richard H. Christ, B.A. (King's Coll.) Measurements Engineer, Radio Corp. of America.
Lt. Leland A. Chvatal, B.A. (Texas A. & M.) United States Army.
Richard R. Clinite, B.S. (Illinois) Swift & Co.
Richard K. Coburn, M.S. (Washington) Chairman of Dept., Church College of Hawaii; Research Specialist, Lockheed Aircraft Corp.
Samuel P. Cohen, A.M. (Temple) Asst. Professor, Pennsylvania State University.
James E. Cohn, B.S. (U.S. Naval Acad.) Asst. Professor, Arizona State University.
Frank B. Contratto, A.A.S. (DeVry Tech. Inst.) Test Engineer, Hewlett-Packard Co.
William K. Cope, B.S. (Eastern Kentucky S.C.) Grad. Student, University of Kentucky.
Nick Coso, Jr., B.S. (Western Reserve) Teacher, Massillon Board of Education, Ohio.
J. Douglas Cowie, B.S. (New Hampshire) Teaching Asst., Rutgers University.
William A. Crabtree, Jr., M.A. (Austin Peay S.C.) Instr., Austin Peay State College.
Dalton L. Criswell, M.Ed. (West Texas S.C.) Teacher, Andrews Independent School, Texas.
James V. Cummings, Student, Manhattan College.
Mrs. Joyce C. Cundiff, M.S. (Florida) Instr., University of Florida.
Mrs. Georgia M. Cunningham, M.A. (Minnesota) Asst. Professor, Jamestown College.
S. Harris Dalrymple, M.A. (Texas) Staff Member, Los Alamos Scientific Lab.
Russell F. DeCesare, Metropolitan Life Insurance Co.
Jack N. Deeter, B.S. (Stanford) Grad. Student, Stanford University.
Michael G. DeGennaro, Student, Manhattan College.
Mrs. Mildred H. Derrick, M.A. (East Carolina Coll.) Instr., East Carolina College.
William P. Devers, Student, University of Rochester; Calculator, Gleason Works.
John E. Diem, Student, Pennsylvania State University.
Roland DiFranco, B.S. (Fordham Coll.) Teaching Asst., Rutgers University.
Norman R. Dilley, M.S. (Southern California) Head of Dept., Huntington Beach High School, California.
Anthony B. DiLuna, M.Ed. (Boston) Teacher, Roslyn High School, New York.
Clyde P. Donahoe, Jr., Student, Highlands High School, San Antonio, Texas.
Francis Y. Doran, B.A. (Oregon) Teacher, South San Francisco High School, California.
Paul L. Dussere, B.A. (Concordia Coll.) Grad. Student, University of Nebraska.
Jack W. Dutton, M.Ed. (Kent S.U.) Grad. Student, University of Kansas.
Andrew M. Economos, Student, University of Florida.
Norman W. Edmund, President, Edmund Scientific Co., Barrington, New Jersey.
Harold D. Edwards-Davies, B.S. (Acadia) Asst. Teacher, Laurentian High School, Ottawa, Ontario.
Arthur D. Eeles, Student, University of Alberta; Staff Teacher, Samuel Crowther High School, Strathmore, Alberta.
Joseph A. Efird, B.S. (St. Thomas) Instr., University of St. Thomas.
Charles W. Eliason, Jr., B.A. (Columbia) Vice President, Commercial National Bank of Peoria.
Charles W. Engel, Jr., B.S. (Wayne S.U.) Grad. Student, Wayne State University.
Mrs. Bernice Engler, B.A. (Adelphi Coll.) Teacher, Brooklyn Technical High School, New York.
Mrs. Jeanne G. English, B.S. (Douglass Coll.) Teacher, Freehold Regional High School, New Jersey.
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Walter S. Evans, M.A. (Columbia) Head of Dept., George School, Pennsylvania.
Elinor R. Evenchick, Student, University of Maryland.
A. Timothy Ewald, M.A. (Columbia) Instr., Medical College of Virginia.
Richard L. Faber, Student, Massachusetts Institute of Technology.
Richard T. Fallon, M.A. (St. John's) Executive Director, Jets, Michigan State University.
Rev. Donald Faught, M.S. (Michigan) Chairman of Dept., Assumption University of Windsor.
Herbert I. Ferguson, B.S. (Southwestern S.C.) Teacher, Public High School, Pawnee Rock, Kansas.
James C. Ferguson, Student, University of Washington.
Arthur A. Finco, M.A. (Iowa S.T.C.) Instr., Mankato State College.
Oscar Firschein, M.S. (Pittsburgh) Logical Design Scientist, Lockheed Missile Systems Division.
Frances M. Flagg, M.A. (Hunter Coll.) Teacher, Bayside High School, New York.
Alvesta C. Flanagan, M.S. in Ed. (Temple) Teacher, Olney High School, Philadelphia, Pennsylvania.
Mrs. Ellen C. Fleming, M.A. (Columbia T.C.) Asst. Professor, East Carolina College.
Peter R. Flusser, B.A. (Ottawa) Grad. Asst., University of Kansas.
Mrs. Ruth G. Fomby, M.A. (East Texas S.C.) Instr., Texarkana College.
Ralph F. Frankowski, M.S. (DePaul) Instr., Barat College.
Charles E. Franti, B.S. (Michigan) Instr., Suomi College, Michigan College of Mining & Technology.
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Louis R. Fuertsch, M.S. (Pittsburg S.C.) Instr., Eastern Oklahoma A. & M. College.
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Francisco Garriga-Rodriguez, M.S. (Chicago) Asso. Professor, University of Puerto Rico.
Frank Gehring, B.A. (Utah) Teacher, West High School, Salt Lake City, Utah.
John R. Gilbert, Jr., M.A. (California S.P.C.) Instr., California State Polytechnic College.
Robert O. Gilmore, M.A. (Yale) Asso. Professor, Elmira College.
Robert C. Glaser, M.S. (Kansas S.T.C., Emporia) Teacher, Mulvane High School, Kansas.
Louis Gmeindl, Jr., B.S. (Muskingum Coll.) Teacher, Beechwood High School, Ohio.
Rev. Robert C. Goodenow, M.A. (St. Louis) Head of Dept., University of Detroit High School.
William Gordon, B.S. (Temple) Pharmacist and Owner, Gordon Pharmacy, Portsmouth, Virginia.
J. Martin Gorfinkel, B.A. (Reed Coll.) Research Asst., Oregon State Tax Commission.
Ernest H. Greene, M.S. (Mississippi) Asst. Professor, Southwestern at Memphis.
Capt. Harold E. Grenard, M.S. (New

York) Meteorologist, United States Air Force.
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Wayne L. Hedger, Student, Carson-Newman College.
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Kenneth J. Hertz, Student, University of Pennsylvania.
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John A. Hildebrant, B.S. (Oklahoma) Grad. Asst., University of Tennessee.
John R. Hinchman, B.S. (Purdue) Engineer, Structures, Rocketdyne Division.
Ansel C. Holland, Ed.D. (Oklahoma) Asst. Professor, East Carolina College.
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Stanley H. Lipson, Student, Trinity College.
Hector J. Llorens, Actuary Asst., George G. Buck, Consulting Actuary.
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Kuo Hwa Lu, Ph.D. (Minnesota) Asso. Professor, Utah State University.
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Claude Marmasse, Ph.D. (Paris) Instr., Hollins College.
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Charles F. McNally, M.S. (Hofstra Coll.) Teacher, Manhasset High School, New York.
John McNamee, Ph.D. (London) Asso. Professor, University of Alberta.
J. Robert Meachem, M.S.E. (Florida) Head of Dept., Eng. & Tech., Palm Beach Junior College.
James Michelow, M.S. (Washington) Teaching Asst., University of Washington.
Marvin V. Mielke, Student, University of Wisconsin.
Walter E. A. Mierzejewski, A.B. (Harvard) Instr., New Bedford Institute of Technology.
Gordon D. Mock, Ph.D. (Wisconsin) Professor, State University of New York Teachers College, Oswego.
Mrs. Blanche G. Moon, Ed.M. (Oklahoma) Teacher, Bethany High School, Oklahoma.
Pierre Morazain, B.A. (Saint-Laurent Coll.) Grad. Student, University of Montreal.
Katsumi Mori, Stringam Ranch, Nemiscam, Alberta.
Joseph T. Morley, M.A. (St. Lawrence) Chairman of Dept., Long Lake Central School, New York.
Gerald E. Murine, M.Ed. (Kent S.U.) Head of Dept., Solon High School, Ohio.
George Mutrie, Student, St. Michael's College, Vermont.
Sydney Myers, Ed.M (Temple) Chairman of Dept., Central

Bucks High School, Doylestown, Pennsylvania.

Lt. Sam P. Nave, B.A. (Texas A. & M. Coll.) Student Naval Aviator, United States Marine Corps.

Allyn H. Nelson, B.S. (Mankato S.C.) Teacher, Cannon Falls High School, Minnesota.

Judith Ng, A.A. (California) Student, University of California, Berkeley.

Edward J. Noe, M.S. (Wisconsin) Teacher, Brockport Central School, New York.

J. George O'Brien, M.A. (Akron) Head of Dept., Roehm Junior High School, Cleveland, Ohio.

Rev. Harold V. O'Leary, M.A. (Detroit) Head of Dept., St. Mary's Boys' High School, Calgary, Alberta.

Jay M. Pasachoff, Student, Harvard College.

James D. Patterson, M.S. (Chicago) Grad. Asst., University of Kansas; Asso. Physicist, Texaco Research Lab.

Mrs. Lucy E. Periolat, A.B. (Randolph-Macon) Teacher, Fenger High School, Chicago, Illinois.

Donald L. Plank, Student, Trinity College.

Andrew W. Plonsky, M.S.E.E. (M.I.T.) Chairman of Eng. Dept., University of Scranton.

Gerald J. Porter, A.B. (Princeton) Teaching Asst., Cornell University.

Malcolm W. Pownall, A.M. (Pennsylvania) Instr., Colgate University.

F. Peck Prather, M.S. (Iowa) Teacher, Cedar Rapids Community School District, Iowa.

Lennart Rade, Fil. Lic. (Gothenburg) Lecturer, Chalmers University of Technology, Sweden.

Howard Raiffa, Ph.D. (Michigan) Professor, Grad. School of Business Administration, Harvard University.

Ralph A. Raimi, Ph.D. (Michigan) Asso. Professor, University of Rochester.

Octavio C. Ramos, 61 West 106 Street, New York 25, New York.

Per-Jan Ranhoff, M.Ed. (New Hampshire) Instr., Pomfret School, Connecticut.

Louis A. Raphael, M.A. (California, Los Angeles) Programmer, Autonetics.

Charles Reeder, B.A. (Huntingdon Coll.) Printer, Advertiser-Journal, Montgomery, Alabama.

James J. Relihan, Jr., Student, Spring Hill College.

Thomas D. Reynolds, Ph.D. (Duke) Asst. Professor, Duke University.

Charles E. Rickart, Ph.D. (Michigan) Professor, Yale University.

Fred D. Rigby, Ph.D. (Iowa) Director, Math. Sciences Division, Office of Naval Research.

William C. Ripperger, Student, Knox College.

James A. Risenhoover, M.S. (Oklahoma S.U.) Teacher, El Reno Junior College.

Paul M. Rivot, Engineering Aide, Sperry Gyroscope Co.

Norma Robinson, Student, University of California, Berkeley.

Donald A. Rock, A.B. (Northwestern) Personnel Research & Procedures, U.S. Army Dept.

Albert Romano, M.A. (Washington, St. Louis) Asst. Professor, Arizona State University.

Steven I. Rosencrans, Student, Massachusetts Institute of Technology.

Saturnino L. Salas, Ph.D. (Yale) Instr., Yale University.

William A. Salmi, Student, Rockford College.

Joseph F. Santner, M.S. (St. Louis) Asst. Professor, Xavier University.

Jack E. Schlossnagel, M.Lit. (Pittsburgh) Asst. Professor, Grove City College

Hubert H. Schneider, Dr.rer.nat. (Munster) Asst. Professor, University of Nebraska.

Gene Schrader, Student, Kansas State Teachers College.

Richard L. Schulz, M.S. (Florida S.U.) Instr., Western Carolina College.

Robert C. Sessler, B.S. (St. Louis) Grad. Student, St. Louis University.

Arthur G. Settle, M.S. in E.E. (Southern California) Electrical Engineer, The Fluor Corp.

Mohammad Y. Shaikh, M.A. (Cambridge) Teaching Asst., University of Tennessee.

Paul J. Shaver, Student, University of Wisconsin.

Sharad-Chandra S. Shrikhande, Ph.D. (North Carolina) Visiting Asso. Professor, University of North Carolina.

Clarence M. Sidlo, B.S. (Northwestern) Supervisor, Computing Section, Sylvania Electronic Systems.

Mrs. Louise A. Simon, M.A.T. (Tulane) Teaching Asst., Tulane University.

Robert C. Simonsen, Student, Los Angeles Harbor College.

Richard D. Simpson, Student, University of Idaho.

Charles D. Sise, B.S. (Bates Coll.) 12 Wakefield Street, Lewiston, Maine.

Sister M. Anne Cathleen, M.A. (St. Louis) Instr., Marycrest College.

Sister M. Pierre, Ph.D. (St. Mary's School of Theology) Professor Marian College.

Sister Mary Leola, B.A. (Loras Coll.) Teacher, Marycliff High School, Spokane, Washington.

Sister Mary Timothy, M.A. (Indiana S.T.C.) Teacher, Providence High School, Fort Wayne, Indiana.

Sister Saint Augustine, Ph.D. (Catholic) Professor, Mt. Saint Mary Academy, Kenmore, New York.

Raymond A. Slater, Student, Trinity College.

Joseph Sloboda, M.A. (Michigan) Instr., Flint Junior College.

Philip Smedley, Student, McMaster University.

Estella M. Smiley, M.A. (California, Los Angeles) Chairman of Dept., David Starr Jordan Junior High School, Burbank, California.

John Sniedze, 62 Park Terrace West, New York 34, New York.

Josef G. Solomon, B.S. (Oklahoma) Engineer, Radio Corp. of America.

Louis A. Spina, M.A. (Columbia) Chairman of Dept., Oyster Bay High School, New York.

Floyd W. Spraktes, B.S. (Idaho) Grad. Student, University of Idaho.

Sidney F. Star, M.S. in C.E. (C.C.N.Y.) Teacher, Thomas Edison Vocational High School, Queens, New York.

C. L. Steininger, M.A. (Toledo) Head of Dept., Rossford High School, Ohio.

John L. Stephenson, M.D. (Illinois) Biophysicist, National Heart Institute.

Arthur I. Sternhell, M.S. (New York) Staff Asst., Metropolitan Life Insurance Co.

Elmer L. Stone, B.S. in Ed. (Northern Illinois) Teacher, Warren Township High School, Gurnee, Illinois

Susan J. Strand, Student, American University.

Allen F. Strehler, Ph.D. (Wisconsin) Asst. Professor, Carnegie Institute of Technology.

C. Gordon Strong, B.A. (Alberta) Principal, Vegreville High School, Alberta.

Sylvester C. Suda, B.S. (North Dakota) Instr., Casselton Public Schools, North Dakota.

Mrs. Louise N. Sutton, M.A. (New York) Asst. Professor, Delaware State College.

Kenneth L. Swanson, B.S. (Ft. Hays, Kansas S.C.) Instr., Ashland High School, Kansas.

Noah J. Swartz, Jr., Student, Indiana Technical College; Trainee-Time Study Engineer, International Harvester.

David S. Tartakoff, Student, Phillips Exeter Academy.

Joseph P. Tassoney, M.S. (Newark Coll. of Eng.) Asst. Professor, Pennsylvania State University.

John N. Thomas, B.Ph. (Northwestern) Instr., Chicago Technical College.

Robert E. Thomas, M.S. (Kansas S.C., Pittsburg) Senior Engineer, The Martin Company.

Howard Eugene Thompson, Ph.D. (Western Reserve) Chief, Organic Section, Dugway Proving Ground.

Francis L. Thomson, A.A. (California) Student, University of California, Berkeley.

Robert R. Tinney, M.S. (Oklahoma S.U.) Instr., Gulf Coast Community College.

William R. Tomlinson, Jr., B.S. (M.I.T.) Staff Member, Operations Research Office, Johns Hopkins University.

Francis E. Toner, M.S. (Fordham) Chairman of Dept., Uniondale High School, New York.

Henry S. Tropp, M.S. (Indiana) Asst. Professor, Humboldt State College.

Elizabeth M. Troxell, M.Ed. (Pennsylvania S.U.) Head of Dept., Garden Spot High School, New Holland, Pennsylvania.

Shelton J. Turner, B.S. (North Carolina) Mathematician, Hercules Powder Co.

William C. Turner, M.A. (Duke) Asst. Professor, Baylor University.

Rev. Alfons J. Van Der Linden, B.S. (St. Augustin Major Seminary, Holland) Head of Dept., Assumption Preparatory School Worcester, Massachusetts.
Larry L. Wagner, Student, Southwestern Louisiana Institute.
William L. Waltmann, M.S. (Iowa S.C.) Instr., Wartburg College.
James A. Ward, Jr., B.S. (New Mexico S.U.) Grad. Student, University of North Carolina.
Allyn J. Washington, M.S. (Brown) Instr., Dutchess Community College.
Ronald H. Weidner, Statistical Applications Analyst, General Electric Co.
Stuart A. Weisberg, B.S. (Polytech. Inst. of Brooklyn) Systems Analyst, Maynard Lab.
Max L. Weiss, M.A. (Cornell) Instr., Reed College.
Charles W. Wheeler, M.B.S. (Colorado) Instr., Contra Costa Junior College.
Colin White, M.S. (Sydney) Asso. Professor, Public Health, Yale University.
John V. Wiggins, Student, Los Angeles State College.
Donald M. Williams, B.S. (Alberta) Teacher, Ladysmith Junior-Senior High School, British Columbia.
Louis L. Williams, M.A. (Columbia) Asso. Professor, East Carolina College.
Warren C. Willig, M.A. (Columbia) Asso. Professor, San Fernando Valley State College.
Arthur P. Wilson, M.A. (Minnesota) Teacher, St. Paul School System, Minnesota.
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J. Shaylor Woodruff, M.A. (Pennsylvania) Head of Dept., Abington Senior High School, Pennsylvania.
Charlene A. Woody, Student, Hampton Institute.
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ANNOUNCEMENT BY THE NEW SECRETARY

The separation of the offices of Secretary and Treasurer, as recently reported in this MONTHLY, vol. 66, p. 619, has now been completed. This is an announcement regarding the division of responsibilities between the Secretary and the Treasurer, insofar as this concerns the membership of the Association.

The Treasurer is responsible for payment of dues and subscriptions, ordering of publications, changes of address, and applications for membership. All correspondence regarding these matters, therefore, should continue to be sent to Professor Harry M. Gehman at the University of Buffalo, Buffalo 14, New York.

The Secretary is responsible for arrangements of national meetings of the Association, the operations of the Board of Governors, the Executive and Finance Committees and all other committees of the Association. Correspondence regarding these items, consequently, should be directed to Professor Henry L. Alder at the University of California, Davis, California.

Your new Secretary wishes to acknowledge to Professor Gehman his deep gratitude for the great help he has received during the transition of offices. During this period of transition, your new Secretary has gained an appreciation of the most effective and devoted leadership which Professor Gehman has given the Association over so many years and for which the membership is greatly in his debt. The Association should consider it fortunate indeed that Professor Gehman will continue in a position of active leadership.

HENRY L. ALDER, *Secretary*

ITINERARIES OF VISITING LECTURERS, 1959–1960

David Blackwell

Feb. 8–10	University of Kentucky, Lexington, Ky. (J. C. Eaves)
Feb. 11–12	University of the South, Sewanee, Tenn. (Gaston S. Bruton)
Feb. 15–16	Florida A. and M. University, Tallahassee, Fla. (Mrs. L. B. Clarke)
Feb. 17–19	Jackson State College, Jackson, Miss. (Benjamin H. McLemore)
Feb. 22–24	Washington University, St. Louis 5, Mo. (H. M. MacNeille)
Feb. 25–27	University of Oklahoma, Norman, Okla. (Richard V. Andree)

Mar. 7–9	Antioch College, Yellow Springs, Ohio. (Thomas C. Holyoke)
Mar. 10–12	Ohio Wesleyan University, Delaware, Ohio. (Robert L. Wilson)
Mar. 14–15	Wayne State University, Detroit 2, Mich. (Wallace Givens)
Mar. 16–17	Eastern Michigan University, Ypsilanti, Mich. (Robert S. Pate)
Mar. 18	Western Michigan University, Kalamazoo, Mich. (Charles H. Butler)
Mar. 21–22	Emmanuel Missionary College, Berrien Springs, Mich. (E. J. Specht)
Mar. 23–25	Marquette University, Milwaukee 3, Wis. (L. J. Heider)
Mar. 28–29	University of Wisconsin, 3203 North Downer Ave., Milwaukee, Wis. (M. Marden)
Mar. 31–Apr. 1	University of Illinois, Urbana, Illinois (Mahlon M. Day)
Apr. 4	Wisconsin State College, Superior, Wis. (John O. Danielson)
Apr. 5–6	University of Minnesota, Duluth, Minn. (John E. Hafstrom)
Apr. 7–8	Mankato State College, Mankato, Minn. (Warren J. Thomsen)
Apr. 11–12	Carleton College, Northfield, Minn. (Kenneth O. May)
Apr. 13–15	Iowa State Teachers College, Cedar Falls, Iowa. (H. C. Trimble)
Apr. 25–26	Fort Hays Kansas State College, Hays, Kan. (W. Toalson)
Apr. 28–29	Bethel College, North Newton, Kan. (Arnold M. Wedel)
May 2–6	Western Illinois University, Macomb, Ill. (Joseph Stipanowich) and Knox College, Galesburg, Ill. (Rothwell Stephens)
May 9–10	Beloit College, Beloit, Wis. (Ralph C. Huffer)
May 11–12	Bradley University, Peoria, Ill. (Marvin G. Moore)
May 13–14	Illinois Section of the M.A.A., Bloomington, Ill. (Donald E. Myers)
May 16–17	Millikin University, Decatur, Ill. (E. W. Ploenges)
May 18–20	Southern Illinois University, Carbondale, Ill. (W. C. McDaniel)
May 23–25	Southern Illinois University, Alton Residence Center, 2809 College Avenue, Alton, Ill. (Eric A. Sturley)

M. R. Hestenes

Feb. 1–2	Mississippi State College, State College, Miss. (Arthur Ollivier)
Feb. 4–5	University of Alabama, University, Ala. (Julian D. Mancill)
Feb. 8–9	McNeese State College, Lake Charles, La. (Stephen M. Spencer)
Feb. 11–12	Stephen F. Austin State College, Nacogdoches, Tex. (W. I. Layton)
Feb. 15–16	Central State College, Edmond, Okla. (Mrs. Dorothea Meagher)
Feb. 18–19	Wayland College, Plainview, Tex. (Dorothy McCoy)
Feb. 29–Mar. 1	St. Olaf College, Northfield, Minn. (C. S. Carlson)
Mar. 3–4	Gustavus Adolphus College, St. Peter, Minn. (Albert G. Swanson)
Mar. 7–9	Wisconsin State College, Eau Claire, Wis. (Lawrence F. Wahlstrom)
Mar. 10–11	Wisconsin State College, River Falls, Wis. (Lillian Gough)
Mar. 14–15	Macalester College, St. Paul Minn. (E. J. Camp)
Mar. 16–18	Concordia College, Moorhead, Minn. (Sigurd Mundhjeld), Moorhead State College, Moorhead Minn. (Marion V. Smith), North Dakota Agricultural College, Fargo, N. D. (A. Glenn Hill)
Mar. 28–29	Nebraska State Teachers College, Kearney, Neb. (L. M. Larsen)
Mar. 30–31, Apr. 1	University of Nebraska, Lincoln, Neb. (William G. Leavitt)
Apr. 4–5	Loras College, Dubuque, Iowa. (Louis E. Ernsdorff)
Apr. 6–8	Wartburg College, Waverly, Iowa. (John O. Chellevold)
Apr. 11–12	Simpson College, Indianola, Iowa. (Harold A. Heckart)
Apr. 13–14	Graceland College, Lamoni, Iowa. (W. S. Gould)
Apr. 25–27	University of Tulsa, Tulsa, Okla. (Ralph W. Veatch)
Apr. 29	Kansas State College, Pittsburg, Kan. (R. G. Smith)
Apr. 30	Kansas Section of the M.A.A., Pittsburg, Kan. (J. D. Haggard)
May 2–3	Kansas State University, Manhattan, Kan. (R. G. Sanger)
May 4	Washburn University, Topeka, Kan. (Paul Eberhart)

May 5–6	University of Kansas, Lawrence, Kan. (G. Baley Price)
May 9–10	Central Missouri State College, Warrensburg, Mo. (Claude H. Brown)
May 12–13	Hope College, Holland, Mich. (Jay E. Folkert)
May 16–17	Heidelberg College, Tifflin, Ohio. (William F. Steele)
May 18–20	Kent State University, Kent, Ohio. (L. E. Bush)

S. A. Jennings

Nov. 16–18	University of Alaska, College, Alaska. (Francis D. Parker)
Nov. 19–20	Seattle Pacific College, Seattle 99, Wash. (O. Karl Krienke, Jr.)
Nov. 23–25	College of Puget Sound, Tacoma 6, Wash. (Edward G. Goman), Pacific Lutheran College, Tacoma 44, Wash. (Eugene A. Maier)
Nov. 30–Dec. 1	Occidental College, Los Angeles 41, Calif. (Charles W. Seekins)
Dec. 3–4	San Diego State College, San Diego, Calif. (C. B. Bell)
Jan. 18–19	Reed College, Portland 2, Ore. (J. B. Roberts)
Jan. 20–22	Lewis and Clark College, Portland, Ore. (Elvy L. Fredrickson), Portland State College, Portland 1, Ore. (Robert W. Rempfer)
Jan. 23–24	Oregon Council of Teachers of Mathematics, 388 Merrill Court, Eugene, Ore. (Wendell C. Hall)
Jan. 27–29	Southern Oregon College, Ashland, Ore. (Floyd L. Taylor)
Feb. 1–2	Arizona State University, Tempe, Ariz. (Lloyd L. Lowenstein)
Feb. 4–5	New Mexico Institute of Mining and Technology, Socorro, N. Mex. (Rafael Sanchez-Diaz)
Feb. 8–9	Montana State College, Bozeman, Mont. (John W. Hurst)

Tibor Rado

Mar. 14–15	Georgetown University, Washington, D. C. and University of Maryland, College Park, Md.
Mar. 17	University of Delaware, Newark, Del.
Mar. 18–19	Villanova University, Villanova, Pa.
Mar. 21–22	Trenton State College, Trenton, N. J.
Mar. 23–26	Simmons College, Boston, Mass., Boston College, Boston, Mass., Boston University, Boston, Mass.
Mar. 28–29	University of Rochester, Rochester, N. Y.
Mar. 31–Apr. 1–2	University of Buffalo, Buffalo, N. Y. and Canisius College, Buffalo, N. Y.
Apr. 11	Connecticut College, New London, Conn.
Apr. 13–15	Wesleyan University, Middletown, Conn.
Apr. 18–19	University of New Hampshire, Durham, N. H.
Apr. 21–22	Dartmouth College, Hanover, N. H.
Apr. 25–30	University of Scranton, Scranton, Pa., Wilkes College, Wilkes Barre, Pa., Misericordia College, Dallas, Pa.
May 2–3	Hamilton College, Clinton, N. Y.
May 9–10	Washington and Lee University, Lexington, Va.
May 12–13	Virginia State College, Petersburg, Va.
May 16–17	University of Richmond, Richmond, Va.
May 19–20	University of South Carolina, Columbia, S. C.
May 22–25	Emory University, Atlanta, Ga., and Georgia Tech., Atlanta, Ga.

Ernst Snapper

Apr. 27	University of Wisconsin, Madison, Wis. (R. H. Bing)
Apr. 28–30	Case Institute of Technology, Cleveland, Ohio. (S.W. McCuskey)
May 2–3	Muskingham College, New Concord, Ohio. (L. Coleman Knight)
May 4–5	Bowling Green State University, Bowling Green, Ohio. (F. C. Ogg)
May 7	Mount Mary College, Milwaukee, Wis., Wisconsin Section, M.A.A.

May 9–10	University of Miami, Coral Gables, Fla. (Georgia Del Franco)
May 11–13	Florida State University, Tallahassee, Fla. (Thomas L. Wade)
May 16–17	Mississippi Southern College, Hattiesburg, Miss. (B. O. Van Hook)
May 19–20	Oklahoma State University, Stillwater, Okla. (L. Wayne Johnston)

AWARD OF THE 1960 CHAUVENET PRIZE

The 1960 Chauvenet Prize will be awarded to Professor Cornelius Lanczos of the Dublin Institute for Advanced Studies for his paper entitled "Linear Systems in Self-Adjoint Form," published in the MONTHLY, Vol. 65 (1958) pp. 665–679. The prize will be presented to Professor Lanczos at the Annual Meeting of the Association to be held at the Hotel Conrad Hilton in Chicago on January 29, 1960.

This is the twelfth award of the Chauvenet Prize since its institution by the MAA in 1925. The regulations governing the award and the names of previous winners appeared in this MONTHLY, vol. 66, 1959, pp. 446–447.

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960

One of a series

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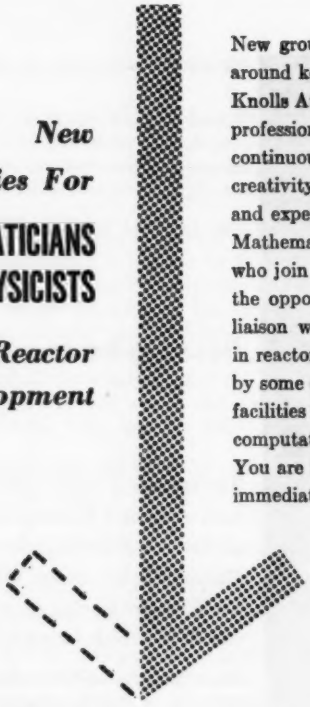

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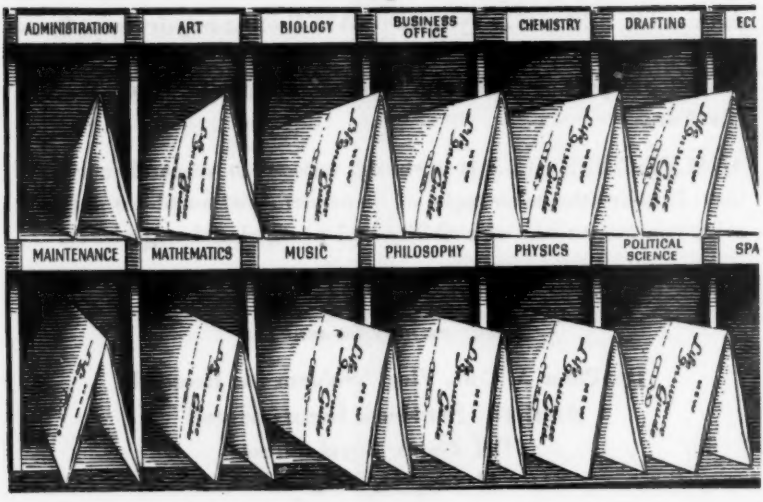

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By PATRICK SUPPES
Stanford University

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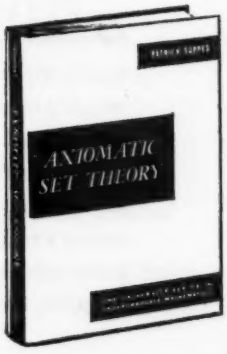

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5 OUTSTANDING MATHEMATICS TEXTS

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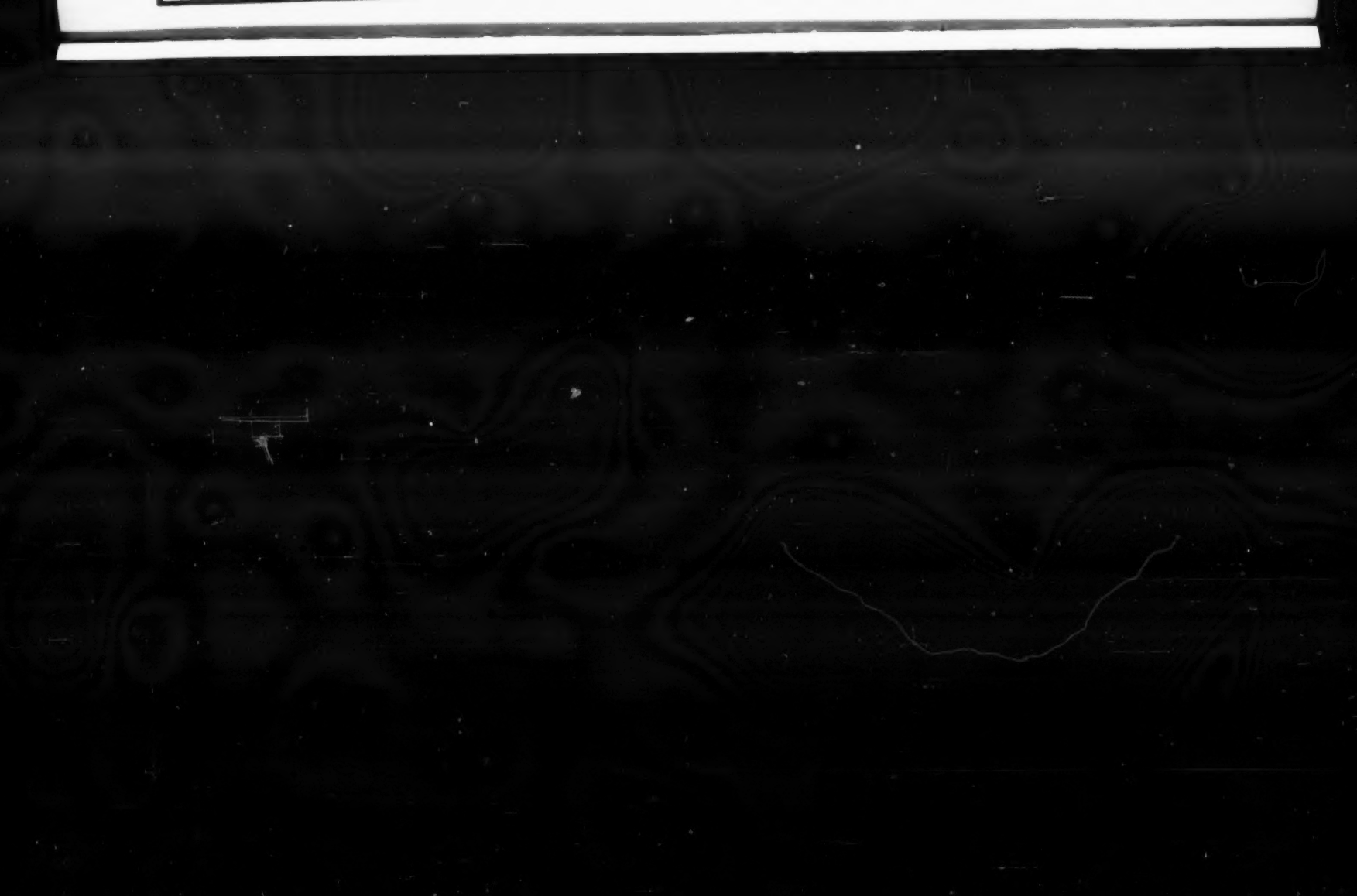

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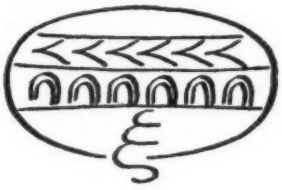

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